Herdable Systems Over Signed, Directed Graphs

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***Abstract*— In this paper, we consider the notion of herdability, a set-based reachability condition, which asks whether the state of a system can be controlled to be element-wise larger than a non-negative threshold. The basic theory of herdable systems is presented and a necessary and sufficient condition for complete herdability is presented. This paper then considers the impact of the underlying graph structure of a linear system on the herdability of the system, for the case where the graph is represented as signed and directed. By classifying nodes based on the length and sign of walks from an input, we find a class of completely herdable systems as well as provide a complete characterization of nodes that can be herded in systems with an underlying graph that is a directed out-branching rooted at a single input. It is shown that for these out-branching systems, it is always possible to herd at least as many nodes as it is possible to control.**

I. Introduction

Controllability is a fundamental property of a dynamical system, and has been an area of study since the work of Kalman et. al in the 1960s [1]. However there are certainly cases where a system need not be completely controllable to achieve desirable system outcomes. Often these systems are studied in the context of stabilizability [2]. This paper considers these systems in a different light by considering the reachability of a specific set rather than the whole state space as in complete controllability. As an example, consider the case where the state of a dynamical system represents the percentage of a given community that will vote for a political candidate and the control input represents advertising. Here an advertising campaign is successful if the state can be driven high enough for the candidate to win, regardless of whether communities can be made to vote at any specific percentage as would be required by complete controllability.

In order to study systems that are not completely controllable but for which certain desirable control outcomes are still achievable, this paper introduces a set-based reachability condition known as herdability, which considers whether the components of the state can be driven above a non-negative threshold. This target set describes desired behavior in social and biological sciences where many systems act based on thresholds, for example collective social behavior [3] and the firing of a neuron [4]. More formally, a continuous time, linear system,

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, is *completely herdable* if there exists a control input that makes the state enter the set $\mathcal{H}_d = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}_i \geq d\}$ for all $d \geq 0$ and where $\mathbf{x}_i$ is the i-th element of $\mathbf{x}$. In the case where a system is not completely herdable, one can consider which elements of the state, $\mathbf{x}$, can be made to reach $\mathcal{H}_d$, which will be referred to as subset herdability. Returning to the example of voting in an election, where $\mathbf{x}_i$ now represents the percentage of community i that will vote for a candidate, one can see that to win the election requires reaching the set $\mathcal{H}_{.5} = \{\mathbf{x} \in \mathbb{R}^n : (\mathbf{x})_i \geq .5\}$.

This paper considers the herdability of linear systems based on the structure of the underlying interaction graph, which encodes information about how states and inputs interact with each other, based on the system matrices A and B. The relationship between a graph and a dynamical system has been previously considered using two primary approaches. The first approach takes classes of dynamical systems and maps them to a graph to discuss properties of all systems that share the same graph structure. This approach is known as structural controllability, [5]–[7] and its extension, strong structural controllability [8]. In structural controllability, a dynamical system is represented by a graph in which each edge of the graph is assigned a weight in $\mathbb{R}$. A system is structurally controllable (strongly structurally controllable) if and only if it is controllable for almost all (all) weights that are assigned to the edges, which is a property that can be verified directly from the structure of the underlying graph.

The second approach goes from a specific graph structure to a system dynamic. In many cases, the system dynamic is consensus dynamics, which are used in robotic and social systems [9], [10]. The controllability of these consensus system has been shown to be directly related to the structure of the graph, either in the ability to identify certain structures [11]–[16] or because the underlying graph is assumed to be of a certain form [17]–[19].

This paper shares the approach of structural controllability in that the control properties of classes of systems are considered based on their graph structure; however in contrast to structural controllability, the graph structures considered here are assumed to be signed graphs. Specifically this paper represents the interaction structure as a signed, directed graph as it often the case that the sign of the interaction structure is sufficient information to determine whether a system is herdable or not. An example is shown in Figure 1. Signed graphs are used in the social networks context to represent systems in which agents are both friends and enemies [20]. In this light, the central problem of the paper can be phrased in a social networks context as follows: how does the grouping

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of friends and enemies in the network relate to the ability to convince agents in the system to hold an opinion in $\mathcal{H}_d$?

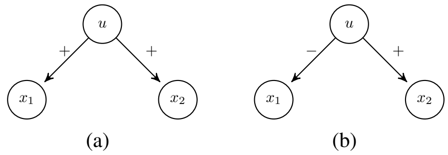

Fig. 1: 1a shows a graph structure which represents a set of systems that are completely herdable, while the systems represented by 1b are not.

The rest of the paper is organized as follows: Section II introduces the basic theory of herdable system. In Section III a graph theoretic characterization of the interaction structure of a linear system is presented. Section IV considers a necessary condition for complete herdability based on the underlying graph structure. Section V presents a class of completely herdable systems and Section VI considers selecting a herdable subset for graphs that are represented by a directed out-branching. The paper concludes in Section VII.

Notation:

For a vector $\mathbf{k} \in \mathbb{R}^n$, $\mathbf{k}_i$ refers to the i-th element of $\mathbf{k}$. For a matrix $K \in \mathbb{R}^{n\times m}$, $(K)_{i,:}$ refers to the i-th row of K, $(K)_{:,j}$ refers to the j-th column of K and $(K)_{i,j}$ to the i,j-th element of K. The cardinality of the set $\mathcal{S}$ is expressed as $|S|$. Let $\mathrm{sgn}(\cdot)$ denote the sign function which is defined as

$$\mathrm{sgn}(x) = \begin{cases} -1 & \text{for } x < 0, \\ 0 & \text{for } x = 0, \\ 1 & \text{for } x > 0. \end{cases}$$

Let $\mathbf{0}_n \in \mathbb{R}^n$ be a vector of zeros, $\mathbf{1}_n \in \mathbb{R}^n$ be a vector of ones, and $0_{n\times m} \in \mathbb{R}^{n\times m}$ be a matrix of zeros. Logical AND is denoted by $\wedge$ and $\vee$ denotes logical OR and $\veebar$ denotes logical EXCLUSIVE OR.

II. Characterizing Herdability

In this section, the basic theory of the herdability of continuous time, linear dynamical systems is presented as well as a characterization of herdability based on the system controllability matrix. Of course before characterizing herdability, the following definitions of herdability are required.

Definition 1: The state i of a linear system is herdable if $\forall \mathbf{x}(0) \in \mathbb{R}^n$ and $h \geq 0$, there exists a finite time t_f and an input $\mathbf{u}(t),\ t \in [0, t_f]$ such that $\mathbf{x}(t_f)_i \geq h$ under control input $\mathbf{u}(t)$.

Definition 2: A set of states, $\mathcal{X} \subseteq \{1, 2, \ldots, n\}$, is herdable if each individual state in $\mathcal{X}$ is herdable together, i.e. if $\forall \mathbf{x}(0) \in \mathbb{R}^n$ and $h \geq 0$, there exists a finite time t_f and an input $\mathbf{u}(t),\ t \in [0, t_f]$ such that $\mathbf{x}(t_f)_i \geq h, \forall i \in \mathcal{X}$ under control input $\mathbf{u}(t)$.

Definition 3: A linear system is completely herdable if all states in the system are herdable together.

To translate the definition of herdability to a necessary and sufficient condition for herdability requires some basic concepts from the study of linear systems, specifically the reachable subspace and the controllability matrix.

Define the reachable subspace $\mathcal{R}[0,t]$ as

$$\mathcal{R}[0,t] = \left\{ \mathbf{x}_1 \in \mathbb{R}^n : \exists \mathbf{u}(\cdot), \mathbf{x}_1 = \int_0^t e^{A(t-\tau)} B \mathbf{u}(\tau) d\tau \right\}.$$

The controllability matrix $\mathcal{C}$ of a linear system is

$$\mathcal{C} = \left[B, AB, A^2B, \ldots, A^{n-1}B\right]$$

It is possible to characterize the herdability of a system based on its controllability matrix. Recall the following from [2] (though any introductory linear systems text will do):

Lemma 1: Theorem 11.5 from [2]

$$\mathcal{R}[0,t] = \mathrm{range}(\mathcal{C}).$$

With Lemma 1 it is possible to prove the following Theorem, which gives a necessary and sufficient condition for the herdability of a subset of states.

Theorem 1: A set of states $\mathcal{X} \subseteq \{1, 2, \ldots, n\}$ is herdable if and only if there is exists a vector $\mathbf{k} \in \mathrm{range}(\mathcal{C})$ that satisfies $\mathbf{k}_i > 0$ for all $i \in \mathcal{X}$.

Proof: Define the set $\mathcal{K}$ to be the set that contains the positive elements of $\mathbf{k}$, $\mathcal{K} = \{p \mid p > 0 \wedge \exists\, i \text{ such that } \mathbf{k}_i = p\}$.

($\mathbf{k} \in \mathrm{range}(\mathcal{C}) \Rightarrow \mathcal{X}$ is herdable) Consider the problem of controlling all states in the set $\mathcal{X}$ to be greater than some lower threshold $h \geq 0$ from an initial condition $\mathbf{x}(0)$. Suppose there is a $\mathbf{k} \in \mathrm{range}(\mathcal{C})$, that satisfies $\mathbf{k}_i > 0$ if $i \in \mathcal{X}$. As $\mathbf{k} \in \mathrm{range}(\mathcal{C})$, $\exists \boldsymbol{\alpha}$ such that

$$\mathcal{C}\boldsymbol{\alpha} = \mathbf{k}.$$

If

$$\gamma > \frac{\max_j \ (h\mathbf{1}_n - e^{At}\mathbf{x}(0))_j}{\min \mathcal{K}}$$

and $\mathbf{v} = \gamma\boldsymbol{\alpha}$ then for all $i \in \mathcal{X}$ it holds that

$$(\mathcal{C}\mathbf{v})_i > (h\mathbf{1}_n - e^{At}\mathbf{x}(0))_i.$$

As the range of $\mathcal{C}$ is the same as the reachable subspace, $\exists \mathbf{u}(\cdot)$ such that for all $i \in \mathcal{X}$

$$(e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)} B\mathbf{u}(\tau)d\tau)_i > h$$

then all states in $\mathcal{X}$ can be made larger that h and as h is arbitrary the subset of states $\mathcal{X}$ is herdable.

($\mathcal{X}$ is herdable $\Rightarrow \mathbf{k} \in \mathrm{range}(\mathcal{C})$) As the set of state nodes $\mathcal{X}$ is herdable, each element of $\mathcal{X}$ can be made larger than some $h^* > 0$ from any initial condition. Consider the initial condition $x(0) = \mathbf{0}_n$. Then by the herdability of the set $\mathcal{X}$ there exists a vector $\mathbf{k}^*$ that satisfies $\mathbf{k}^*_i > h^*\ \forall i \in \mathcal{X}$ and an input $\mathbf{u}(\cdot)$ such that

$$\int_0^t e^{A(t-\tau)} B\mathbf{u}(\tau)d\tau = \mathbf{k}^*$$

Then $\mathbf{k}^*_i > 0\ \forall i \in \mathcal{X}$ by the definition of h^*. By the definition of $\mathcal{R}[0,t]$, $\mathbf{k}^* \in \mathcal{R}[0,t]$ and consequently $\mathbf{k}^* \in \mathrm{range}(\mathcal{C})$ by Lemma 1. ■

Corollary 1: A linear system is completely herdable if and only if there exists an element-wise positive vector $\mathbf{k} \in \mathrm{range}(\mathcal{C})$.

III. Characterizing Dynamical Systems via Graphs

The dynamical system in Equation (1) can be represented by three graphs; each of which contains different levels of information about the interactions between the states and inputs. The first is an unweighted directed graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the vertex (or node) set and $\mathcal{E}$ is the edge set. This graph is commonly used in the study of structural controllability [5] and will not be explored further in this paper. The second graph is a signed graph $G^s = (\mathcal{V}, \mathcal{E}, s(\cdot))$ where $s(\cdot)$ accepts an edge and returns a label in $\{+1, -1\}$, which is the sign of the edge. This signed graph represents a class of systems whose edge weights have the same sign pattern. The third graph is a weighted graph $G^w = (\mathcal{V}, \mathcal{E}, w(\cdot))$ where $w(\cdot)$ accepts an edge and returns a weight in $\mathbb{R}$. The weighted graph is the representation of a single system.

As will be seen later, the weighted graph G^w can be directly related to the controllability grammian and therefore the controllability properties of the system. This paper focuses on the interplay between G^s and G^w, in that the presented structural results are cases where the results for the herdability of a system based on the weighted G^w can be extended to all signed graphs with the same sign structure G^s regardless of the weights of the edges in G^w, a notion similar to strong structural controllability [8].

The formal definition of the graphs follows. The set of vertices satisfies $\mathcal{V} = \mathcal{V}_x \cup \mathcal{V}_u,\ \mathcal{V}_x \cap \mathcal{V}_u = \emptyset$, where $\mathcal{V}_x = \{v_{x1}, v_{x2}, \ldots, v_{xn}\}$ is a set of vertices representing the states of the system and $\mathcal{V}_u = \{v_{u1}, v_{u2}, \ldots, v_{um}\}$ is a set of nodes representing the inputs to the system. An arbitrary element of $\mathcal{V}$ will be referred to by v_i for some index i, as will arbitrary elements $v_{xi} \in \mathcal{V}_x$ and $v_{ui} \in \mathcal{V}_u$. The state i will now be interchangeably referred to by the node v_{xi} as will the input j and the node v_{uj}.

The edge set satisfies $\mathcal{E} = \mathcal{E}_x \cup \mathcal{E}_u$ where the edges in $\mathcal{E}_x$ represent interactions between states of the system, while $\mathcal{E}_u$ represents interactions between the inputs and the states. Denote the directed edge from v_i to v_j as (v_i, v_j). Then $(v_{xi}, v_{xj}) \in \mathcal{E}_x \Leftrightarrow A(j,i) \neq 0$ and $(v_{ui}, v_{xj}) \in \mathcal{E}_u \Leftrightarrow B(j,i) \neq 0$. An arbitrary element of $\mathcal{E}$ will be referred to by e_i for some i. By partitioning the node and edges sets, it is possible to define the state subgraph $G_x = (\mathcal{V}_x, \mathcal{E}_x)$, which captures only interactions between states as well as the input subgraph $G_u = (\mathcal{V}, \mathcal{E}_u)$ which captures interactions from the inputs to the states. Note that the input nodes do not interact with each other nor is it possible to have an edge (v_{xi}, v_{uj}).

When considering the signed graph G^s, $s((v_{xi}, v_{xj})) = \mathrm{sgn}(A(j,i))$ and $s((v_{ui}, v_{xj})) = \mathrm{sgn}(B(j,i))$. Similarly for G^w, $w((v_{xi}, v_{xj})) = A(j,i)$ and $w((v_{ui}, v_{xj})) = B(j,i)$.

As an example, consider the system

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 0 & 0 \\ 5 & 0 & 2 \\ 4 & -3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{u} \tag{2}$$

which is translated into G^s and G^w in Figure 2.

To describe these graphs requires a number of basic definitions from graph theory. Unless otherwise noted, the provided definitions follow [21]. A walk from v_0 to v_p, $\pi(v_0, v_p)$, is any alternating sequence of nodes and edges $\pi(v_0, v_p) = v_0, e_1, v_1, e_2, v_2 \ldots, v_{p-1}, e_p, v_p$ such that $v_i \in \mathcal{V}\ \forall i$ and $e_i = (v_{i-1}, v_i) \in \mathcal{E}$. The set of walks from v_0 to v_p is $\theta(v_0, v_p)$. A node v_j is reachable from v_i, which will be written as $v_i \to v_j$, if $\theta(v_i, v_j) \neq \emptyset$[1]. The length of a walk, $\mathrm{len}(\pi)$, is equal to the number of edges in π.

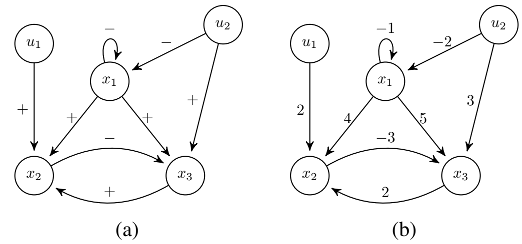

Fig. 2: The graphs of the system in in Equation (2). 2a: G^s the signed graph. 2b: G^w the weighted graph.

A walk has an associated sign which follows

$$s(\pi) = \prod_{e_i \in \pi} s(e_i).$$

In this paper, a walk also has an associated weight:

$$w(\pi) = \prod_{e_i \in \pi} w(e_i).$$

This is distinct from the weight of a walk as it is treated in many applications, such as shortest path algorithms, which consider $w(\pi) = \sum_{e_i \in \pi} w(e_i)$ [22]. Referring back to the example in Figure 2, the walk $\pi(u_1, x_3) = u_1, (u_1, x_2), x_2, (x_2, x_3), x_3$ is of length 2 and has $s(\pi(u_1, x_3)) = -1$ and $w(\pi(u_1, x_3)) = -6$.

To begin classifying the system in Equation (1) based on the signed graph G^s, we define two basic types of sets. Let $\mathcal{N}_d^j$ be the set of nodes reachable from v_{uj} via at least one negative walk of length d. Similarly $\mathcal{P}_d^j$ is the set of nodes reachable from v_{uj} through at least one positive walk of length d. If there is only one input to the system, the superscript will be dropped to refer to $\mathcal{N}_d$ and $\mathcal{P}_d$ instead of $\mathcal{N}_d^1$ and $\mathcal{P}_d^1$. Figure 3 shows an example of these sets.

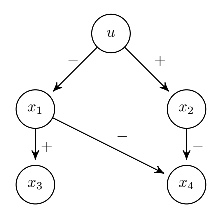

Fig. 3: An example of $\mathcal{N}_d$ and $\mathcal{P}_d$: $\mathcal{N}_1 = \{x_1\}$, $\mathcal{N}_2 = \{x_3, x_4\}$, $\mathcal{P}_1 = \{x_2\}$, $\mathcal{P}_2 = \{x_4\}$

As will be seen, the sets $\mathcal{P}_d^j$ and $\mathcal{N}_d^j$ often provide sufficient information to determine the herdability of a system. To show this requires classifying the structure of the weighted

[1] Reachability is discussed within both graph theory and control theory. This paper will use the term reachable in both senses, with clarification only if it is uncertain which notion of reachability is considered.

graph G^w. Consider the total weight of positively signed walks from input v_{uj} to node v_{xi} with length d,

$$\rho^+_{j\to i,d} = \sum_{\pi\in\theta^+_d(v_{uj},v_{xi})} w(\pi),$$

where $\theta^+_d(v_{uj}, v_{xi})$ is the set of positive walks of length d from v_{uj} to v_{xi}. From the definition of $\mathcal{P}^j_d$, it holds that $\rho^+_{j\to i,d} > 0$ if $v_{xi} \in \mathcal{P}^j_d$ and 0 else. Similarly the total weight of negatively signed walks from input v_{uj} to node v_{xi} with length d is

$$\rho^-_{j\to i,d} = \sum_{\pi\in\theta^-_d(v_{uj},v_{xi})} w(\pi),$$

where $\theta^-_d(v_{uj}, v_{xi})$ is the set of negative walks of length d from v_{uj} to v_{xi} and it follows that $\rho^-_{j\to i,d} < 0$ if $v_{xi} \in \mathcal{N}^j_d$ and 0 else. Then the weight of all walks from input v_{uj} follows:

$$\rho_{j\to i,d} = \rho^+_{j\to i,d} + \rho^-_{j\to i,d}.$$

Consider the example shown in Figure 4.

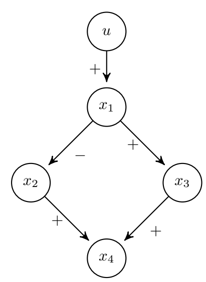

Fig. 4: An example of a signed graph where $\mathcal{N}_d$ and $\mathcal{P}_d$ don't completely determine $\rho_{j\to i,d}$

The signed graph represents all systems of the form

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\alpha_1 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_3 & \alpha_4 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \beta_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{u}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1 > 0$. Here the total walk weight to node v_{x4} at length 2 is

$$\rho_{1\to 4,2} = \beta_1 (\alpha_2\alpha_4 - \alpha_1\alpha_3)$$

which can be positive, negative or zero depending on the values of the various constants. While it is possible to define a notion of structural herdability to capture how $\rho_{1\to 4,2}$ depends on the system constants, this paper focuses instead on the case where $\mathcal{N}^j_d$ and $\mathcal{P}^j_d$ uniquely determine the sign of $\rho_{j\to i,d}$.

The case where the sign of $\rho_{j\to i,d}$ is determined by $\mathcal{N}^j_d$ and $\mathcal{P}^j_d$ is shown in the following Lemmas. These Lemmas follow directly from the definitions of the sets $\mathcal{P}^j_d$ and $\mathcal{N}^j_d$ and as such are presented without proof.

Lemma 2: If $v_{xi} \in \mathcal{P}^j_d \wedge v_{xi} \notin \mathcal{N}^j_d$ then $\rho_{j\to i,d} > 0$.

Lemma 3: If $v_{xi} \in \mathcal{N}^j_d \wedge v_{xi} \notin \mathcal{P}^j_d$ then $\rho_{j\to i,d} < 0$.

It is possible to relate $\rho_{j\to i,d}$ with the system matrices A, B and ultimately the controllability properties of the system. Define a weighted adjacency matrix $\tilde{A}_w$ for G^w_x, where $(\tilde{A}_w)_{i,j} = w((v_{xj}, v_{xi}))$ if $(v_{xj}, v_{xi}) \in \mathcal{E}_x$ and $(\tilde{A}_w)_{i,j} = 0$ if not. Define a weighted adjacency matrix $\tilde{B}_w$ for G^w_u, where $(\tilde{B}_w)_{i,j} = w((v_{uj}, v_{xi}))$ if $(v_{uj}, v_{xi}) \in \mathcal{E}_u$ and $(\tilde{B}_w)_{i,j} = 0$ if not. Note that from the definition of the weight of an edge, $\tilde{A}_w = A$ and $\tilde{B}_w = B$. Then $(A^{d-1}B)_{i,j}$ is the sum of the weight of all walks of length d from v_{uj} to v_{xi}. More formally:

Lemma 4:

$$(A^{d-1}B)_{i,j} = \rho_{j\to i,d}.$$

Proof: The result will be shown via proof by induction on d. Consider the case of $d = 1$. By the definition of the weight of an edge:

$$(B)_{i,j} = \rho_{j\to i,1}.$$

Consider the weight of all walks of length d from an input v_{uj} to a state node v_{xi}. By assumption, $(A^{d-2}B)_{i,j} = \rho_{j\to i,d-1}$. As $A^{d-1}B = AA^{d-2}B$, it follows that

$$(A^{d-1}B)_{i,j} = \sum_{k=1}^{n} (A)_{i,k}\rho_{j\to k,d-1}.$$

As a walk of length d is the concatenation of a walk of length $d-1$ and a walk of length 1, it follows from the definition of the weight of a walk that

$$\sum_{k=1}^{n} (A)_{i,k}\rho_{j\to k,d-1} = \rho_{j\to i,d}.$$

■

As $\mathcal{C}$ is the concatenation of matrix products from B to $A^{n-1}B$, Lemma 4 shows that the herdability of the system in Equation (1) is determined by walks on G^w which have lengths from 1 to n. Further:

Lemma 5: $(\mathcal{C})_{i,(m(d-1)+j)} = \rho_{j\to i,d}$.

Proof: From Lemma 4,

$$(A^{d-1}B)_{i,j} = \rho_{j\to i,d}.$$

From the definition of the controllability matrix, the submatrix

$$(\mathcal{C})_{:,m(d-1)+1:md} = A^{d-1}B.$$

The result follows. ■

IV. A Necessary Condition for Complete Herdability

This section shows how graph structure and system herdability are related by providing a necessary condition for complete herdability of a system known as input connectability. It also explores some examples that show why input connectability is only a necessary condition.

Definition 4: A graph is input connectable if

$$\bigcup_{v_{uj}\in\mathcal{V}_u} \mathscr{R}_j = \mathcal{V}_x,$$

where $\mathscr{R}_j$ is the set of nodes reachable from v_{uj}: $\mathscr{R}_j = \{v_{xi} \in \mathcal{V}_x \mid v_{uj} \to v_{xi}\}$.

To show the necessity of input connectability, requires that the condition on the range of $\mathcal{C}$ presented in Theorem 1 be used to show Lemma 6, which characterizes the herdability of a single node.

Lemma 6: A state i is herdable if and only if $\exists j$ such that

$$(\mathcal{C})_{i,j} \neq 0.$$

Proof: ($(\mathcal{C})_{i,j} \neq 0 \Rightarrow i$ Herdable) If $(\mathcal{C})_{i,j} \neq 0$ then by appropriate choice of the j-th element of a vector $\mathbf{z}$ it holds for a positive constant w that:

$$(\mathcal{C}\mathbf{z})_i = w$$

Then there is a vector $\mathbf{k} \in \mathrm{range}(\mathcal{C})$ with $\mathbf{k}_i > 0$ and v_{xi} is herdable by Theorem 1.

(Herdable $\Rightarrow (\mathcal{C})_{i,j} \neq 0$) Suppose the contrary. Then by assumption $\forall j\ (\mathcal{C})_{i,j} = 0$. Consider making $\mathbf{x}(t) \geq h$ from an initial state $\mathbf{x}(0) = \mathbf{0}_n$. As $\forall j\ (\mathcal{C})_{i,j} = 0$, it holds that $\forall z \in \mathrm{range}(\mathcal{C}),\ z_i = 0$ and by Lemma 1 for any reachable $\mathbf{x}(t)\ \forall t \geq 0$, $\mathbf{x}(t)_i = 0$ and state i is not herdable. ■

If a single node is not herdable then the system is not completely herdable. As such, Lemma 6 can be used to show the following.

Theorem 2: If a system is completely herdable, then it is input connectable.

Proof: Suppose not. Then by assumption, there exists a node v_{xi} such that $v_{xi} \notin \bigcup_j \mathscr{R}_j$ and as such there is no walk from an input to v_{xi}. If there is no walk to v_{xi}, then $(\mathcal{C})_{i,:} = \mathbf{0}_n$ by Lemma 5 and the node will not be herdable by Lemma 6. As such, the system is not completely herdable. ■

Consider the following two examples that show why input connectability is only a necessary condition and not a sufficient condition. These examples are presented here as the condition of Theorem 5 in Section V ensures that the system is input connectable and that the cases presented in these examples do not occur.

The first example has to do with the structure of the signed graph G^s. We return to the example given in the Section I, which is shown again in Figure 5.

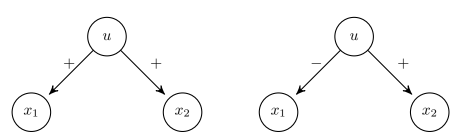

Fig. 5: The systems represented by the graph structure in 5a are completely herdable, while 5b shows a graph structure that is never completely herdable.

Figure 5a represents systems of the form

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \mathbf{u}$$

where $\beta_1, \beta_2 > 0$, which gives a controllability matrix:

$$\mathcal{C} = \begin{bmatrix} \beta_1 & 0 \\ \beta_2 & 0 \end{bmatrix}$$

And by inspection,

$$\mathrm{range}(\mathcal{C}) = \mathit{span}\left(\left\{\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}\right\}\right).$$

This system is always completely herdable.

On the other hand, Figure 5b can be translated to systems of the form:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -\beta_1 \\ \beta_2 \end{bmatrix} \mathbf{u}$$

where $\beta_1, \beta_2 > 0$. This gives a controllability matrix:

$$\mathcal{C} = \begin{bmatrix} -\beta_1 & 0 \\ \beta_2 & 0 \end{bmatrix}$$

And by inspection,

$$\mathrm{range}(\mathcal{C}) = \mathit{span}\left(\left\{\begin{bmatrix} -\beta_1 \\ \beta_2 \end{bmatrix}\right\}\right).$$

Here either v_{x1} or v_{x2} can be made larger than all thresholds $h \geq 0$ but not both. This example illustrates a fundamental trade off when herding signed digraphs, which is that at a given distance from the input either $\mathcal{N}_d$ or $\mathcal{P}_d$ can be herded but not both. In the language of social networks, it is not possible to simultaneously convince an enemy and a friend.

It turns out that Figure 5a is an example of a positive system. In the case of a positive system, input connectability is a necessary and sufficient condition for complete herdability. Before showing this, consider the following definitions.

A system is positive if an element-wise non-negative initial state under element-wise non-negative control input remains element-wise non-negative [23]. Further a positive system is excitable if and only if each state variable can be made positive by applying an appropriate nonnegative input to the system initially at rest [$\mathbf{x}(0) = \mathbf{0}_n$] [23].

With these definitions, it is possible to re-prove the theorem of [24] in light of the characterization of Theorem 1:

Theorem 3: A positive linear system is completely herdable if it is input connectable.

Proof: By Theorem 8 of [23], an input connectable, positive linear system is excitable. Then there is an element-wise positive vector in the reachable subspace, which is also the range of the controllability matrix by Lemma 1. Then by Corollary 1, the system is completely herdable. ■

The second example that shows why input connectability is only a necessary condition and not a sufficient condition can be seen based on the weighted graph G^w, specifically the cancellation of walk weights from an input to a state node. It is possible that a node be included in both $\mathcal{N}_d^j$ and $\mathcal{P}_d^j$ which could lead to a combination of weights such that $\rho_{j\to i,d} = 0$. If the only walks to v_{xi} are of length d then the node v_{xi} is not herdable, as is the case for v_{x4} in Figure 4. The following lemma shows a condition which ensures this undesirable interaction does not occur.

Lemma 7: If $v_{xk} \in \mathcal{N}_d^j \cup \mathcal{P}_d^j \wedge v_{xk} \notin \mathcal{N}_d^j \cap \mathcal{P}_d^j$ then $\rho_{j\to i,d} \neq 0$.

Proof: Suppose the contrary. Then

$$\rho_{j\to k,d} = 0$$
$$\rho^+_{j\to k,d} + \rho^-_{j\to k,d} = 0.$$

As $v_{xk} \in \mathcal{N}_d^j \cup \mathcal{P}_d^j$ it holds that

$$\rho^+_{j\to k,d} > 0,\ \rho^-_{j\to k,d} < 0$$

which implies that

$$v_{xi} \in \mathcal{P}_d^j,\ v_{xi} \in \mathcal{N}_d^j$$
$$v_{xi} \in \mathcal{P}_d^j \cap \mathcal{N}_d^j$$

■

V. A Class of Completely Herdable Systems

This section draws together the definitions and concepts from the previous sections to describe a class of completely herdable systems, characterized both by the controllability matrix and the structure of the underlying graph.

Theorem 4: If for each state $i \in \{1, 2, \dots, n\}$, there exists a distance d and an input $j \le m$ such that $(\mathcal{C})_{i,m(d-1)+j} \neq 0$ and each element of the vector $(\mathcal{C})_{:,m(d-1)+j}$ has the same sign then the system is completely herdable.

Proof: For state i let d^i be the distance and let j^i be the input, which satisfy the conditions of the Theorem and define $\gamma_i = m(d^i - 1) + j^i$. For ease of exposition, define the sign of a column when all elements of the column have a single sign so that $sign((\mathcal{C})_{:,\gamma_i}) = 1$ if $sign((\mathcal{C})_{i,\gamma_i}) = 1$ and $sign((\mathcal{C})_{:,\gamma_i}) = -1$ if $sign((\mathcal{C})_{i,\gamma_i}) = -1$.

Consider $\Gamma = \bigcup_i \gamma_i$, the set of all γ_i such that there is no repeated values. This is necessary as it may be that for two states i and j, $\gamma_i = \gamma_j$.

Construct the vector $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha}_\kappa = 0$ if $\kappa \notin \Gamma$, $\boldsymbol{\alpha}_\kappa = 1$ if $\kappa \in \Gamma$ and $sign((\mathcal{C})_{:,\kappa}) = 1$ and $\boldsymbol{\alpha}_\kappa = -1$ if $\kappa \in \Gamma$ and $sign((\mathcal{C})_{:,\kappa}) = -1$.

As the condition of the Theorem holds for all $i \in \{1, 2, \dots, n\}$, there exists $\mathbf{k} \in \mathbb{R}^n$ which is element wise positive such that

$$\mathcal{C}\boldsymbol{\alpha} = \mathbf{k}$$

and the system is completely herdable by Corollary 1. ■

The following Theorem provides a case where the composition of the sets $\mathcal{P}_d^j$ and $\mathcal{N}_d^j$ uniquely determines the herdability of the graph.

Theorem 5: If for each $v_{xi} \in \mathcal{V}_x$, there exists a distance d and an input v_{uj} such that $v_{xi} \in \mathcal{N}_d^j \cup \mathcal{P}_d^j$ and $\mathcal{N}_d^j = \emptyset \veebar \mathcal{P}_d^j = \emptyset$ then the system is completely herdable.

Proof: Consider the herdability of a node v_{xi} which satisfies $v_{xi} \in \mathcal{N}_{d^i}^{j^i} \cup \mathcal{P}_{d^i}^{j^i}$ and $\mathcal{N}_{d^i}^{j^i} = \emptyset \veebar \mathcal{P}_{d^i}^{j^i} = \emptyset$ for some d^i and v_{uj^i}. As $\mathcal{N}_{d^i}^{j^i} = \emptyset \veebar \mathcal{P}_{d^i}^{j^i} = \emptyset$, it must be that $v_{xi} \in \mathcal{N}_{d^i}^{j^i} \cup \mathcal{P}_{d^i}^{j^i}$ and $v_{xi} \notin \mathcal{N}_{d^i}^{j^i} \cap \mathcal{P}_{d^i}^{j^i}$ as $\mathcal{N}_{d^i}^{j^i} \cap \mathcal{P}_{d^i}^{j^i} = \emptyset$. From Lemma 5 and Lemma 7, this implies $(\mathcal{C})_{i,m(d^i-1)+j^i} \neq 0$. Additionally, as $\mathcal{N}_{d^i}^{j^i} = \emptyset \veebar \mathcal{P}_{d^i}^{j^i} = \emptyset$, Lemma 2 and Lemma 3 show that all nonzero elements of $(\mathcal{C})_{:,m(d^i-1)+j^i}$ have the same sign. As this hold for all v_{xi}, the system is completely herdable by Theorem 4. ■

As Theorem 5 only provides a sufficient condition for herdability there are cases where the condition of Theorem 5 does not hold but the system is still completely herdable. Figure 6 shows a simple example.

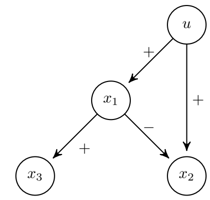

Fig. 6: An example of a completely herdable graph which does not satisfy the condition of Theorem 5.

VI. Subset Selection: Directed Out-branchings

If a system is not completely herdable, it is still possible to control a subset of the system nodes to enter the set $\mathcal{H}_d$. This section presents such a selection procedure in the special case of graphs that are a rooted out-branching.

A directed graph, $\hat{\mathcal{G}} = (\hat{\mathcal{V}}, \hat{\mathcal{E}})$ is a rooted out-branching if it has a root node $v_i \in \hat{\mathcal{V}}$ such that for every other node $v_j \in \hat{\mathcal{V}}$ there is a single directed walk from v_i to v_j. The case considered here is that of a single input, input rooted out-branching, which means that every node $v_{xi} \in \hat{\mathcal{V}}_x$ has a single in-bound walk from the single input v_u. The unique walk from v_u to v_{xi} in the input-rooted out-branching will be referred to as $\pi_t(v_u, v_{xi})$. Consider the maximum walk length between v_u and a state node, which is

$$d_{\max} = \max_{v_{xi} \in \hat{\mathcal{V}}_x} \operatorname{len}(\pi_t(v_u, v_{xi})).$$

Let $\mathbb{H}_u$ be the set of nodes made larger than some lower threshold $h \ge 0$ via a signal from the input v_u.

Theorem 6: In an input rooted, out-branching, $\mathbb{H}_u$ follows

$$\mathbb{H}_u = \bigcup_{d=1}^{d_{\max}} \mathcal{X}_d,$$

where $\mathcal{X}_d \in \{\mathcal{P}_d, \mathcal{N}_d, \emptyset\}$.

Proof: Consider the ability to herd a node v_{xi} and assume that $\operatorname{len}(\pi_t(v_u, v_{xi})) = d_i$. As there is only one walk from v_u to v_{xi} it holds that $(\mathcal{C})_{i,d} = 0,\ \forall d \in \mathcal{D},$ such that $d \neq d_i$ and $(\mathcal{C})_{i,d_i} \neq 0$. Further v_{xi} is either in $\mathcal{P}_d$ or in $\mathcal{N}_d$ but can not be in both as there is only one path to v_{xi}. Then if v_{xi} is in $\mathcal{P}_{d_i}$, $\rho_{u\to i,d} > 0$ by Lemma 2 and consequently $(\mathcal{C})_{i,d_i} > 0$ by Lemma 5 or if v_{xi} is in $\mathcal{N}_{d_i}$, $\rho_{u\to i,d} < 0$ by Lemma 3 and $(\mathcal{C})_{i,d_i} < 0$ by Lemma 5.

Then it follows that $(\mathcal{C})_{:,d_i}$ uniquely determines the ability to herd all nodes at distance d_i. If $\alpha_{di} = 1$ then $((\mathcal{C})_{:,d_i}\alpha_{di})_i > 0,\ \forall i$ such that $v_{xi} \in \mathcal{P}_{d_i}$ and $\mathcal{P}_{d_i}$ is herdable by Theorem 1. If $\alpha_{di} = -1$ then $((\mathcal{C})_{:,d_i}\alpha_{di})_j > 0,\ \forall i$ such that $v_{xi} \in \mathcal{N}_{d_i}$ and $\mathcal{N}_{d_i}$ is herdable by Theorem 1. Finally if $\alpha_{di} = 0$ then $(\mathcal{C})_{:,d_i}\alpha_{di} = \mathbf{0}_n$ and no nodes are herded. Then by the appropriate choice of α_{di} the set of nodes that can be herded at distance d_i from u, $\mathcal{X}_{di}$ must be one of $\{\mathcal{P}_d, \mathcal{N}_d, \emptyset\}$.

Construct a vector $\boldsymbol{\alpha} \in \mathbb{R}^n$ where $\forall d \in \{1, 2, \dots, d_{\max}\}$

$$\boldsymbol{\alpha}_d = \begin{cases} 1 & \text{so that } \mathcal{X}_d = \mathcal{P}_d, \\ -1 & \text{so that } \mathcal{X}_d = \mathcal{N}_d, \\ 0 & \text{so that } \mathcal{X}_d = \emptyset, \end{cases}$$

and where the remaining $n - d_{\max}$ elements are 0. Then $\mathcal{C}\boldsymbol{\alpha}$ shows the herdability of the set of nodes $\bigcup_{d=1}^{d_{\max}} \mathcal{X}_d$. ■

Corollary 2: The maximal collection of nodes, $\mathbb{H}_u^*$, that can be herded in a input rooted out-branching satisfies

$$|\mathbb{H}_u^*| = \sum_{l=1}^{d_{\max}} \max(|\mathcal{N}_l|, |\mathcal{P}_l|).$$

In the case of an single input, input connectable, directed out-branching where $\forall d \in \{1, 2, \ldots, d_{\max}\}$, $\mathcal{N}_d = \emptyset \veebar \mathcal{P}_d = \emptyset$, Corollary 2 shows that $|\mathbb{H}_u^*| = n$, or equivalently that the system is completely herdable. Figure 7 shows an example of selecting the set of nodes that can be herded in an input rooted, out-branching.

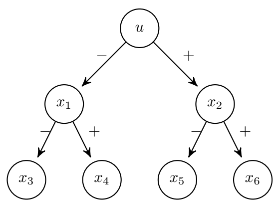

Fig. 7: An example of an input rooted out-branching

The graph in Figure 7 can be translated into the following class of systems:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha_1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_3 & 0 & 0 & 0 & 0 \\ 0 & \alpha_4 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -\beta_1 \\ \beta_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{u}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2 > 0$. The system has a controllability matrix:

$$\mathcal{C} = \begin{bmatrix} -\beta_1 & 0 & 0 & 0 & 0 & 0 \\ \beta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1\beta_1 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_2\beta_1 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_3\beta_2 & 0 & 0 & 0 & 0 \\ 0 & \alpha_4\beta_2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$\text{range}(\mathcal{C}) = span \left(\left\{ \begin{bmatrix} -\beta_1 \\ \beta_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \alpha_1\beta_1 \\ -\alpha_2\beta_1 \\ -\alpha_3\beta_2 \\ \alpha_4\beta_2 \end{bmatrix} \right\} \right)$$

As such the possible sets of herded nodes are $\{1,3,6\}, \{1,4,5\}, \{2,3,6\}, \{2,4,5\}$.

The result of Theorem 6 is similar in nature to the k-walk controllability theory put forward in [25]. The k-walk theory shows that for each $d \in \{1, 2, \ldots, d_{\max}\}$ one element of either $\mathcal{N}_d$ or $\mathcal{P}_d$ can be controlled. In the graph given in Figure 7, the possible sets of nodes that can be controlled are $\{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{2,3\}, \{2,4\}\{2,5\}, \{2,6\}$. As a consequence of the k-walk theory, the maximal collection of nodes that are controlled in a directed out-branching from input v_u, $\mathbb{C}_u^*$, satisfies

$$|\mathbb{C}_u^*| = d_{\max}.$$

In the case of herding a network, Corollary 2 shows that the maximal collection of nodes, $\mathbb{H}_u^*$, will satisfy

$$d_{\max} \leq |\mathbb{H}_u^*| \leq n.$$

Therefore in the worst case, the same number of nodes can be herded as can be controlled and depending on the network structure many more nodes can be herded. Note that the results of Theorem 6 do not extend directly to the multi-input out-branching case, as in a multiple input out-branching the sets $\mathcal{P}_d^j$ and $\mathcal{N}_d^j$ no longer uniquely determine the ability to herd a node.

VII. Conclusion

In this paper, we present a characterization of the herdability of a subset of the state space via a condition on the range of the controllability matrix, $\mathcal{C}$. A classification of the underlying system graph allowed for the exploration of a number of consequences of the condition on the range of $\mathcal{C}$ including that input connectability is a necessary condition for complete herdability and that the sets $\mathcal{P}_d^j$ and $\mathcal{N}_d^j$ can be used to characterize both a class of completely herdable systems and the nodes that can be herded in a single input, input rooted out-branching.

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