# Herding Complex Networks

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***Abstract*—The problem of controlling complex networks is of interest to disciplines ranging from biology to swarm robotics. However, controllability can be too strict a condition, failing to capture a range of desirable behaviors. Herdability, which describes the ability to drive a system to a specific set in the state space, was recently introduced as an alternative notion. This paper considers the application of herdability to the study of complex networks, first under the assumption that a positive system evolves on the network, and then in the general case. In the positive systems case, the herdability of a class of networked systems is investigated and two problems related to ensuring herdability are explored. The first is the input addition problem, which investigates which nodes in a network should receive inputs to ensure that the system is herdable. The second is a related problem of selecting the best single node from which to herd the network, in the case that a single node is guaranteed to make the system herdable. A novel control energy-based herdability centrality measure is introduced in order to select the best herding node. In the general case, a previously introduced method for testing whether a system is completely herdable based on the underlying sign pattern of the system matrices is compared to a novel optimization based framework on a set of signed complex networks.**

## I. Introduction

Controlling networked systems has long been of interest to the controls community [1]–[3] and has recently received considerable attention from the complex networks community [4]. Complete controllability is often used to describe the ability of a complex network to be controlled, however many systems do not require complete controllability for desired system behavior to be achieved. This paper considers instead an alternative notion called herdability, which describes systems that are not completely controllable but for which a class of desirable behaviors are still possible [5], [6].

Herdability is particularly applicable to understanding the behavior of complex networks such as social and biological systems. A system is completely herdable if all the elements of the state can be brought above a threshold by the application of a control input. Thresholds capture an important class of behavior in biological and social systems, in which a system reaches a tipping point and as a result the behavior of a system may change dramatically. Examples of behavior driven by thresholds include the firing of a neuron [7], quorum sensing in bacteria [8] and collective social action [9], [10].

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Selecting nodes to ensure a system is herdable is an example of an input addition problem [11]. Input addition problems have been previously considered in the case of controllability of complex networks. In multi-agent systems, this problem is referred to as the leader selection problem [12], [13], which determines the controllability of a system following consensus dynamics based on a given selection of leader nodes. The input addition problem has also been considered when seeking to ensure system controllability of a system that does not necessarily follow consensus dynamics [11], [14]–[17]. In the case of known dynamics, finding the minimum number of state nodes to actuate to ensure controllability was shown to be NP-hard [15]. A similar problem, that of selecting nodes to ensure reachability to a specific end point or subspace, was also found to be NP-hard [18]. In contrast to these results, this paper shows that in the case of a known, positive system it is possible to determine in linear time which nodes will ensure that the system is herdable when input is applied to them.

The input addition problem has also been considered for the case of a structured system [19], a modeling framework that attempts to capture the behavior of a class of systems that share the same underlying interaction pattern, by applying the notion of structural controllability [20]–[22]. A system is structurally controllable if for almost all collections of edge weights the system is controllable. In the case of structured systems, it has been shown that input selection can be done efficiently [14], [16].

Ensuring the herdability of the system when the underlying network representation has a fixed sign pattern is also considered here, which is related to sign controllability [23]–[26]. Sign controllability, which determines controllability of a class of systems that share the same underlying sign pattern, builds on two sets of results from the economics and ecology literature. The first is sign stability [27], [28], which asks whether a matrix is stable based on its sign pattern. The other is sign solvability [29]; which asks, when solving the matrix equation $A\mathbf{x} = \mathbf{b}$, whether the sign pattern of $\mathbf{x}$ is uniquely determined by the sign pattern of $A$ and $\mathbf{b}$. While sign controllability has been considered with regard to the structure of the various system matrices [23]–[25] as well as the sign pattern of the underlying graph [26], it has yet to be applied to complex network data.

This paper considers the problem of characterizing nodes based on control energy, which has been previously considered in the context of controllability. A number of controllability based centralities were introduced in [30] and [31], some of which were extended to include considerations of robustness to noise in [32] and used to study the brain in [33]. This paper also considers control energy; however the formulation of the control problem here is based on the notion of herdability which is guaranteed to hold for a single input node in the systems considered. There is no similar guarantee for controllability that makes comparison to controllability related metrics difficult.

The rest of the paper is organized as follows: In Section II the basic definitions of herdable systems are introduced. Section III considers the problem of selecting nodes to ensure a positive system is herdable. In Section IV, a novel centrality measure is introduced to compare nodes in a herdable network. Section V considers the general case of selecting nodes to herd a network based on sign pattern. The paper concludes in Section VI.

## II. Herdable Systems

A networked system can be described by its graph structure and the dynamics that act over the graph structure. Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{E}$ is the directed edge set and $\mathcal{V} = \mathcal{V}_x \cup \mathcal{V}_u$, where $\mathcal{V}_x$ is the set of state nodes and $\mathcal{V}_u$ is the set of control nodes, which together satisfy $\mathcal{V}_x \cap \mathcal{V}_u = \emptyset$. Let $\|\mathcal{V}_u\| = m$ and $\|\mathcal{V}_x\| = n$, where $\|\cdot\|$ denotes cardinality. Each state node $v_i \in \mathcal{V}_x$ has an associated scalar state $x_i$ and each control node $\mu_i \in \mathcal{V}_u$ has a scalar input $u_i$. The interaction dynamics of the system are assumed to be linear:

$$\dot{x} = Ax + Bu,$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x = [x_1, x_2, \ldots, x_n]^T$, and $u = [u_1, u_2 \ldots, u_m]^T$. The structure of the system matrices $(A, B)$ of the linear system are derived from the underlying graph $\mathcal{G}$. In general a non-zero element of $A$, $a_{ij}$, corresponds to the weight on an edge leaving from state node $v_j \in \mathcal{V}_x$ and entering state node $v_i \in \mathcal{V}_x$. Similarly a non-zero element of $B$, $b_{ij}$, corresponds to an edge from input node $u_j \in \mathcal{V}_u$ to state node $x_i \in \mathcal{V}_x$.

Herdability considers the general problem of going from any initial point in the state space, $x(0)$, to a terminal set [5], [6]. Specifically, the terminal set, $\mathcal{H}_d$, is a shifted positive orthant, defined as $\mathcal{H}_d = \{x \in \mathbb{R}^n : x_i \geq d\}$. The following definition characterizes the complete herdability of a system.

**Definition 1.** *A system is completely herdable if* $\forall x(0) \in \mathbb{R}^n$ *and* $\forall d \geq 0$ *there exists a finite time* $T$ *and an input* $u(t),\ t \in [0, T]$ *s.t.* $x(T) \in \mathcal{H}_d$ *under control input* $u(t)$.

This paper first considers that the dynamics evolving over the network correspond to consensus dynamics, which are an example of a positive system. A continuous time system is positive if the weights on edges between nodes in a network are positive. Self edges are allowed to be negative. The study of positive systems covers a large range of complex networks, including subject areas ranging from epidemic spread and, more generally, compartmental systems in biology to consensus in opinion dynamics and robotics [34]. As shown in [5], the herdability of a positive system can be characterized based on its underlying graph structure.

**Theorem 1** (Theorem 4 from [5])**.** *A positive linear system is completely herdable if and only if it is input connectable, i.e. there is a path from an input to any state node in the underlying graph structure.*

Input connectability is a necessary condition for structural controllability [11], sign controllability [26] as well as for herdability [5]. This paper considers the implication of the above Theorem for the application of herdability to a positive system. The positive system assumption is reasonable in both the control context, where a significant portion of the work on control of networked systems discusses consensus formulations which are positive systems [2] (though that has been changing since [35]), and in the complex systems context, as many complex network representations are either unweighted or have positive weights between edges.

In the case that the underlying network representation has both positive and negative edges between nodes, determining whether a given input renders the system herdable requires a more in depth analysis. In [6], conditions that ensure herdability were found based on the sign pattern of the controllability matrix $\mathcal{C} = [B, AB, A^2B, \ldots, A^{n-1}B]$ as well as the sign pattern of the underlying network topology. These conditions and their application in complex networks will be considered further in Section V.

Before concluding the introduction to herdable systems, let us consider how herdability differs from controllability; specifically with regard to symmetry with respect to an input. The two major lines of work on the controllability of networks, that of structural controllability and consensus dynamics over networks, have identified symmetry with respect to a control input as an sufficient condition for the loss of controllability of a system [12], [20].

Symmetric nodes must be controlled together, which violates the condition of complete controllability. As herdability looks only at driving the state to be larger than some threshold, the herdability condition is satisfied even when the symmetric nodes are controlled to the same point. An illustrative case of symmetric systems is the star or hub graph, shown in Figure 1. The fact that symmetry degrades controllability explains why past analysis of controllability of complex networks has found that driver node selection avoids hubs [14]. In the case of herdability, it is possible to select the center of the star to apply input to herd a positive system as in Figure 1b. In the case of a signed network, fewer nodes than in the case of controllability can receive input to ensure herdability, with the specifics depending on the underlying sign pattern as in Figure 1c.

## III. Selecting Herding Nodes

It is often the case when interacting with networked systems that instead of being given an existing set of interconnections with input nodes, the problem is one of selecting the state nodes with which to interact to ensure the system is herdable, i.e. to design the $B$ matrix of the linear system. To this end, this section considers the input selection problem: how to select a *minimal* subset, $\mathbb{H}$, consisting of $N_H$ state nodes that ensures herdability of the system, where each element of $\mathbb{H}$ is called a herding node. Note that based on the desired terminal set and the assumption of a positive system, once $\mathbb{H}$ is identified,

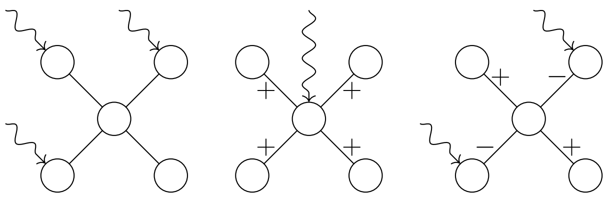



Fig. 1: 1a: Structural controllability analysis will select 3 nodes in order to ensure controllability of the system. 1b: If the system is positive, any node can be selected to ensure that the system is herdable, including the middle node, as symmetry does not degrade the ability to herd the network. 1c: If there are both positive and negative weights in the network, ensuring that the system is herdable can require applying input to multiple nodes.

system herdability can be ensured with a $B \in \mathbb{R}^n$ equals 1 at position $i$ if $i \in \mathbb{H}$ and 0 otherwise.

Consider now the problem of making a given system herdable by adding input to make a network input connectable. The solution to this problem will be called a *Herding Cover*. In order to generate a herding cover, the system must first be decomposed into strongly connected components (SCCs). This can be achieved in linear time by Kosaraju's algorithm [36]. Once the strongly connected components are identified, a graph condensation is performed which generates a directed acyclic representation of the graph $\mathcal{G}$, represented as $\mathcal{G}_a = (\mathcal{V}_a, \mathcal{E}_a)$. Each element of $\mathcal{V}_a$ represents a strongly connected component of $\mathcal{G}$ and an edge is in $\mathcal{E}_a$ if there is a link in $\mathcal{E}$ between any nodes in the respective SCCs [37]. Let $N_r$ be the number of roots of this acyclic representation.

**Theorem 2.** *It holds that*

$$N_H = N_r.$$

*As such, $N_H$ can be determined in linear time.*

*Proof.* Consider the acyclic representation of the graph with adjacency matrix $A$. Each root node of this graph represents a SCC of the graph that has no in-bound edges from other SCCs. By applying input to one node from each such SCC, then by the definition of strong connectivity the entire SCC is herdable as well as all nodes downstream from the given SCC. As input is applied to all roots, the entire system is herdable.

This spanning forest representation can be computed in linear time with respect to the original network size. The roots of this forest representation can found in linear time, by checking each node in $\mathcal{V}_a$ to find the nodes with zero in-degree. □

**Corollary 1.** *If the graph is undirected or consists of disjoint strongly connected components, then*

$$N_H = N_w,$$

*where $N_w$ is the number of weakly connected components.*

**Corollary 2.** *If the directed graph $\mathcal{G}$ is strongly connected, then any one node set forms the root of a Herding Cover.*

Table I shows results for analysis of the fraction of herding nodes, $n_H$, compared with the fraction of driver nodes, $n_c$, from the controllability analysis of [14]. Across all considered networks $n_H \leq n_c$. In 15 of the 24 networks, herdability requires communication with fewer nodes than controlling the network as $n_H < n_c$. There are some networks, such as the Western US Power Grid, where $n_H << n_c$. These networks consist of a single SCC, which can be made herdable with one herding node as shown in Corollary 2.

## IV. Herdability Centrality

If the system is herdable from any one node, a secondary issue arises of selecting which one node to use as the herding node. To select between nodes in a SCC, a new herdability centrality measure is proposed that takes into account the energy required to drive the system into the set $\mathcal{H}_d$. While many networks are not necessarily strongly connected, as mentioned previously any directed graph can be broken down into a non-overlapping set of SCCs in linear time; allowing each SCC to be considered individually to determine the herdability centrality.

Consider the problem of entering $\mathcal{H}_d$ from the origin with minimal control energy:

$$\begin{aligned} J(B,d) = \min_{u(t)} \int_0^{t_f} & \|u(\tau)\|^2 d\tau \\ \text{s.t. } & \dot{x}(t) = Ax(t) + Bu(t),\ t \in [0, t_f] \\ & x(t_f) \in \mathcal{H}_d \\ & x(0) = 0_n, \end{aligned} \tag{1}$$

where the minimum energy, $J$, is parameterized by the structure of the interaction with the control inputs, given in the matrix $B$, and by $d > 0$ which is assumed to be fixed.

The formulation in Eq. (1) can be contrasted with the minimum energy optimal control problem as typically studied, i.e. in the context of completely controllable systems. Specifically the desired end position of the system is typically a desired final point $x_f$ instead of the set $\mathcal{H}$. In general, for systems that are not completely controllable, there is no guarantee that a desired $x_f$ or even $\mathcal{H}$ can be reached. However if the system is herdable, then by definition the reachable subspace from $0_n$, which we denote $R(0)$, intersects the set $\mathcal{H}_d$.

**Theorem 3.** *If the system is herdable, then the minimum energy to reach $\mathcal{H}_d$ is of the form*

$$x_f^T W_c^+ x_f,$$

*where $x_f \in \mathcal{H}_d \cap R(0)$, and $W_c^+$ is the Moore-Penrose pseudo-inverse of the controllability grammian:*

$$W_c = \int_0^{t_f} e^{A\tau} B B^T e^{A^T \tau} d\tau.$$

*Proof.* If the network is herdable then $\exists x_f \in \mathcal{H}_d \cap R(0)$. This reachable $x_f$ allows the use of a number of properties of the controllability grammian. To reach $\forall x_f \in R(0) \cap \mathcal{H}_d$ requires an input $u(t)$ that satisfies $\int_0^t e^{A(t-\tau)} Bu(\tau) d\tau = x_f$. This $u(t)$ will have the form $u(t) = B^T e^{At} p$ where $W_c p = x_f$.

| Type | Name | N | L | Dir. | $n_w$ | $n_H$ | $n_c$ |
|---|---|---|---|---|---|---|---|
| Collaboration | Astro-Physics [38] | 16,706 | 242,502 | U | 1 | 0.062 | 0.080 |
| | Condensed Matter Physics [38] | 16,726 | 95,188 | U | 1 | 0.071 | 0.108 |
| | Cond. Mat. Physics 2003 [38] | 31,163 | 240,058 | U | 1 | 0.051 | 0.090 |
| | Cond. Mat. Physics 2005 [38] | 40,421 | 351,384 | U | 1 | 0.045 | 0.083 |
| | High Energy Physics [38] | 8,361 | 31,502 | U | 1 | 0.159 | 0.208 |
| | Network Science [39] | 1,589 | 5,484 | U | 1 | 0.249 | 0.260 |
| | Jazz [40] | 198 | 5,484 | U | 1 | 0.005 | 0.005 |
| | General Relativity [41] | 26,196 | 28,980 | U | 1 | 0.813 | 0.816 |
| Biological | C. Elegans Neural [42] | 306 | 2,345 | D | 3.7 | 0.121 | 0.190 |
| | Protein Interaction [43] | 2,114 | 4,480 | U | 1 | 0.197 | 0.462 |
| | Dolphin Social [44] | 62 | 318 | U | 1 | 0.016 | 0.032 |
| Infrastructure | Western US Power Grid [42] | 4,941 | 13,188 | U | 1 | 0.0002 | 0.116 |
| | Top Airports [45] | 500 | 5960 | U | 1 | 0.002 | 0.250 |
| | Football Games [46] | 115 | 1,226 | U | 1 | 0.009 | 0.009 |
| Online | UCIonline [47] | 1,899 | 20,296 | D | 138 | 0.291 | 0.323 |
| | Political Blogs [48] | 1,490 | 19,025 | D | 1.89 | 0.340 | 0.471 |
| Friendship | Third Grade [49] | 22 | 177 | D | 1 | 0.046 | 0.046 |
| | Fourth Grade [49] | 24 | 161 | D | 1 | 0.042 | 0.042 |
| | Fifth Grade [49] | 22 | 103 | D | 1 | 0.046 | 0.046 |
| | Highschool [50] | 73 | 243 | D | 2 | 0.137 | 0.178 |
| | Fraternity [51] | 58 | 1,934 | U | 1 | 0.017 | 0.017 |
| | EIES 1 [52] | 32 | 650 | D | 1 | 0.031 | 0.031 |
| | EIES 2 [52] | 32 | 759 | D | 1 | 0.031 | 0.031 |
| | Mine [53] | 15 | 88 | U | 1 | 0.067 | 0.067 |

TABLE I: For each network, the table shows the number of nodes $N$, the number of edges $L$, whether the network is Undirected or Directed, the ratio of number of herding nodes to number of weakly connected components $n_w = \frac{N_H}{N_w}$, the fraction of herding nodes $n_H = \frac{N_H}{N}$, the fraction of driver nodes $n_c = \frac{N_c}{N}$.

There exists a solution to $W_c p = x_f$ as $R(0) = \text{range}(W_c)$ i.e. that $x_f \in \text{range}(W_c)$. These solutions are of the form

$$p^* = W_c^+ x_f + [I - W_c^+ W_c] x_f$$

with $p^* = W_c^+ x_f$ as the unique solution in the range of $W_c$, where $W_c^+$ can here refer to any generalized inverse [54]. If $W_c^+$ refers specifically to the Moore Penrose Inverse (or any generalized reflexive inverse) the form of the minimum energy to reach $x_f$ is $x_f W_c^+ x_f$. □

With the analytical expression for the minimum energy to reach $x_f$, it is possible to reframe Eq. (1) as the problem of choosing the optimal $x_f$ in the set $\mathcal{H}_d \cap R(0)$:

$$\begin{aligned} \min_{x_f} \; & x_f^T W_c^+ x_f \\ \text{s.t. } & x_f \geq d \\ & x_f \in R(0) \\ & x(0) = 0_n. \end{aligned}$$

Here the problem can once again be simplified further based on properties of the controllability grammian. As $W_c$ is a symmetric, real matrix, the eigenvectors of $W_c$ are mutually orthogonal and the eigenvectors with non-zero eigenvalues span the range of $W_c$ [55]. When $\text{rank}(W_c) = r \leq n$ there are $r$ eigenvectors $\{v_1, \ldots, v_r\}$ associated with the $r$ non-zero eigenvalues $\lambda_1, \ldots, \lambda_r$ which form an orthonormal basis for $\text{range}(W_c)$. Therefore as $x_f \in \text{range}(W_c)$

$$x_f = \sum_{i=1}^{r} \alpha_i v_i. \tag{2}$$

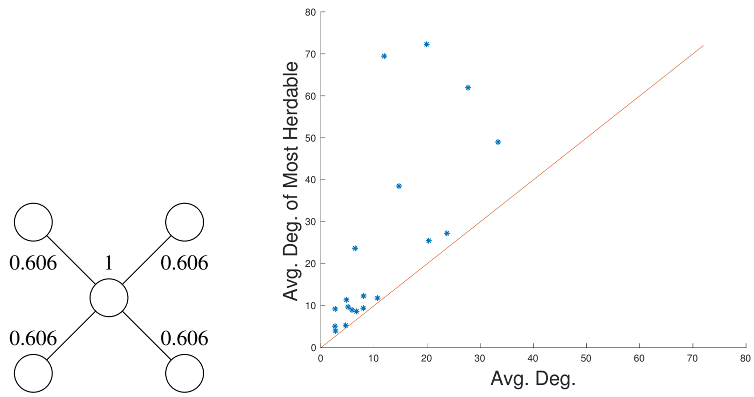
Using that $v_i$ are orthonormal and also eigenvectors of $W_c^+$ with associated eigenvalues $\frac{1}{\lambda_i}$, substituting in Eq. (2) gives

$$\begin{aligned} \min_{\alpha} \; & \sum_{i=1}^{r} \frac{\alpha_i^2}{\lambda_i} \\ \text{s.t. } & V\alpha \geq d, \end{aligned} \tag{3}$$

where $V = [v_1 \ldots v_r]$. The problem in Eq. (3) can be more efficiently solved than that in Eq. (1), allowing larger networks to be analyzed. This formulation also shows the similarity to the measure known as average controllability [31], which is defined as $\text{trace}(W_c^{-1})$. However as many complex networks are close to uncontrollable, this metric is not explicitly calculated [33].

### A. Calculating Herdability Centrality

With a simplified version of the minimum energy optimal control problem in hand, it is possible to move on to calculating herdability centrality. Each state node of the herdable system is considered in turn as the sole input node allowing the




(a) (b)

Fig. 2: Herdability Centrality and Hubs: (a) Herdability centrality of a star graph. The center of the star requires the lowest input energy to reach $\mathcal{H}$ and hence has the highest herdability centrality. (b) Plot of average degree of the complete network vs average degree of the top $10\%$ most herdable nodes, with a line representing average network degree.

calculation of $J_i = J(e_i, d)$, where $e_i \in \mathbb{R}^n$ is 1 at position $i$ and 0 elsewhere, and $d > 0$ is fixed. The quantity $J_i$ is the minimum energy to reach $\mathcal{H}$ using only node $i$ as control input. In order to compare the minimum energy across nodes, the herdability centrality for node $i$, $Hc_i$, is defined as

$$Hc_i = \frac{\min_k\{J_k\}}{J_i}.$$

Herdability centrality is normalized to be between 0 and 1, which aids interpretability as the control energy for complex networks can be quite large. As reaching $\mathcal{H}$ with minimum energy is the chosen metric when interacting with these networks, the node(s) with minimum energy to reach $\mathcal{H}$ across all nodes will have the highest herdability centrality.

For the purpose of calculating herdability centrality of existing complex networks, the largest SCC of each considered network is used as the underlying interaction topology. The dynamics are assumed to be a modification of consensus dynamics, related to the opinion dynamic model of Taylor, which captures the effect of an external source of information on the opinion of an agent [56]. When node $i$ is the sole herding node, the consensus dynamics are as follows:

$$\dot{x}_j(t) = \sum_{z \in \mathcal{N}_j} (x_z(t) - x_j(t)), \quad \forall j \neq i$$
$$\dot{x}_i(t) = \sum_{k \in \mathcal{N}_i} (x_k(t) - x_i(t)) + u(t) - x_i(t),$$

where $\mathcal{N}_i$ is the set of nodes with edges entering node $i$. In order to improve efficiency of the calculation, the final time is taken to be $t_f = \infty$ as the infinite horizon controllability grammian can be solved for efficiently, if $A$ is stable, as the solution to the continuous time Lyapunov equation $AW_c + W_cA + BB^T = 0$. Note that while consensus does not normally provide a stable $A$, the model above does.

As mentioned previously, herdability allows hubs to be selected to herd complex systems, though it is not known a priori that hubs will indeed be selected. Figure 2(a) shows that the center node of the hub has the highest herdability centrality, and therefore requires the least energy to reach $\mathcal{H}_d$. Figure 2(b) shows that the introduced herdability centrality tends to select nodes that have higher than average degree, i.e. that herdability centrality tends to select hubs.

### B. Comparison to Other Centrality Measures

Given that herdability centrality tends to select high degree nodes, the question becomes whether it is possible to forgo the computationally expensive herdability centrality calculation in favor of an inexpensive graph structure based calculation. Table II introduces a number of centrality measures which will be compared against herdability centrality.

| Name | Description |
|---|---|
| In-Degree Centrality | The number of in-bound edges |
| Eccentricity | The maximum distance from the node to any other node |
| Closeness Centrality | The sum of the reciprocal of the distance to each other nodes |
| Betweenness Centrality | The number of shortest paths that pass through the node divided by the total number of shortest paths between two nodes |
| Eigenvalue Centrality | For node $i$, the $i$th component of the dominant eigenvector of the Adjacency Matrix |
| Katz Centrality [57] | The weighted sum of all paths, where a path of length $d$ receives a weight of $\alpha^d, \alpha > 0$. |

TABLE II: Description of Centrality Measures

Figure 3 shows that while high herdability centrality nodes tend to have high degree, the highest in-degree node does not necessarily have high herdability centrality. This holds for all centrality measures considered. In 8 of the 19 networks considered the traditional centrality measures overlap with the highest herdability centrality nodes. However, there is no single centrality measure which can be used reliably to select the minimum energy herding node. The overlap between herdability centrality and existing measures tends to occur in undirected networks. As control energy is related to the symmetry structure of the network [13], it may be that, in undirected networks, the existing centrality measures

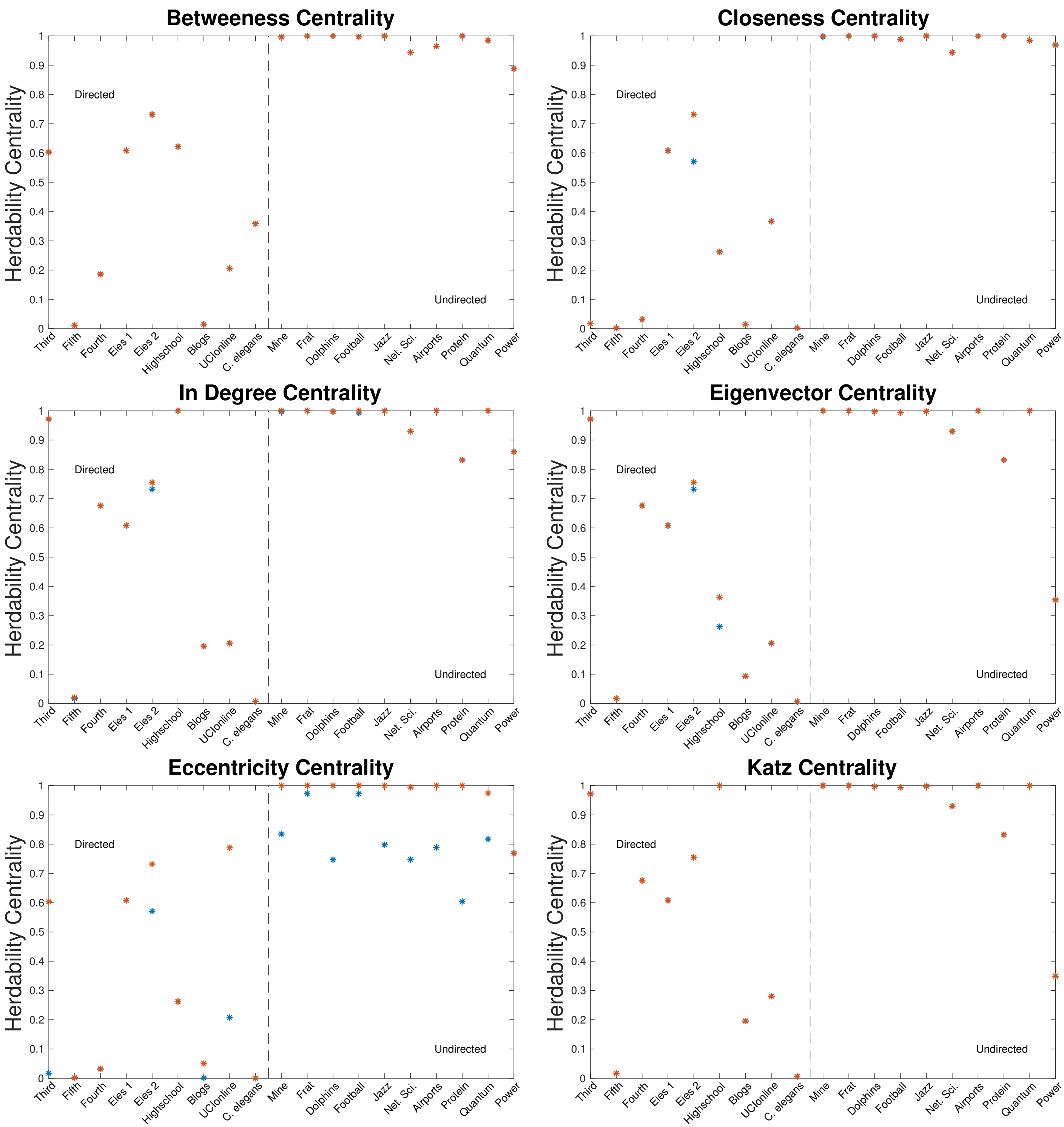



Fig. 3: Selecting the Highest Herdability Node: Each subgraph considers a different centrality measure and shows the highest (in red) and lowest (in blue if present) herdability centrality of the node(s) identified as having the highest value for each respective centrality Within each categorization (directed or undirected) the networks are organized starting with the smallest network on the left. In all undirected networks, all calculated centrality measures have high herdability centrality. In some directed networks, In-Degree and Katz centrality identify high herdability nodes.

provide information about the symmetry structure. Examining the directed networks shows that size of the network seems to have no impact on overlap. For example, in the Fifth Grade Friendship network, $N = 22$, all considered centrality measures select a node with low herdability centrality.

## V. Signed Networks

Having considered in-depth the input selection problem in the case of a positive system, we now consider the more general case of herdability when edges between nodes are allowed to have both positive and negative weights, as would happen when there is antagonism in the underlying social network. This section provides an initial estimate of how herdable complex networks are by selecting single nodes in the network as input nodes and then calculating the resulting number of herded nodes.

As the starting point to determine the number of nodes that can be herded, we apply the results of [6] of which the key theorems are shown below. The first two are necessary and sufficient conditions on the range space of $\mathcal{C}$, the controllability matrix, and $W_c$, the controllability grammian.

**Theorem 4** (Theorem 1 in [6])**.** *A subset of states $\mathcal{X} \subseteq \{x_1, x_2, \dots, x_n\}$ in a linear system is herdable if and only if there is exists a vector $\mathbf{k} \in \text{range}(\mathcal{C})$ that satisfies $\mathbf{k}_i > 0$ for all $x_i \in \mathcal{X}$.*

**Theorem 5** (Corollary 1 in [6])**.** *A set of states $\mathcal{X} \subseteq \{x_1, x_2, \dots, x_n\}$ in a linear system is herdable if and only if there is exists a vector $\mathbf{k} \in \text{range}(W_c)$ that satisfies $\mathbf{k}_i > 0$ for all $x_i \in \mathcal{X}$.*

The second is a necessary condition based on the matrix $[A\ B]$.

**Theorem 6** (Theorem 2 in [6])**.** *If a linear system is completely herdable then there exists an element-wise positive vector $\mathbf{k} \in \text{range}([A\ B])$.*

The above theorems provide an avenue to test whether a given linear system $\dot{x} = Ax + Bu$ is herdable by first using $A$ and $B$ to calculate the required matrix $\mathcal{M} \in \{\mathcal{C}, W_c, [A\ B]\}$ (depending on which theorem is used) and then checking the range of the desired matrix. Note that verifying that an element wise positive vector is in the range of a matrix can be difficult for large systems. Methods to perform this verification will be discussed in depth later in this section.

Before discussing how to translate the above theorems into methods for verifying that a system is herdable, it's worth discussing the properties of the three matrices $\{\mathcal{C}, W_c, [A\ B]\}$. As the eventual goal is to determine herdability for large scale systems we are concerned with the computational cost of producing the desired matrix. We begin with the least computationally intensive matrix to produce, $[A\ B]$. While the test based on $[A\ B]$ is attractive as it requires negligible computational cost to generate the matrix, the condition of Theorem 6 is only a necessary condition, which means that should an element wise positive vector be found in the range of $[A\ B]$, another method would be required to make a definitive statement that the system is herdable. For the systems that are considered later in this section, the condition on the range of $[A\ B]$ is treated as an upper bound, capturing the largest number of node that could be expected to be herded from a given node.

The controllability grammian is an attractive alternative and has been used when discussing controllability of complex networks, especially brain networks [33]. In the case of large scale networks, the infinite horizon controllability grammian is used as it can computed in linear time. However $A$ must be stable for the infinite horizon controllability grammian to be computed. For the networks analyzed below, it was assumed that $A = \mathcal{A}(\mathcal{G})$, that the matrix $A$ is the adjacency matrix of the graph. This was done in order to best match the methodology of [6] which considers a one to one correspondence between the $A$ matrix and the graph which is analyzed. In this case, the resulting system is unstable and the infinite horizon controllability grammian can not be computed.

Finally Theorem 4 can be used to show herdability based on the controllability matrix $\mathcal{C}$, which can be computationally expensive and the resulting matrix that must be analyzed can be large, depending on the number of inputs, $m$, as $\mathcal{C} \in \mathbb{R}^{n \times nm}$. However, the controllability matrix does have the advantage that if the complex network is structurally balanced[1] then the controllability matrix test gives information about the herdability of a class of systems that share the same sign pattern [6]. Additionally if the system is unstable, the controllability matrix can be partially computed, which gives some information about the herdability of the system.

Having discussed the matrices that one can use to test herdability, we now turn to the question of how one tests herdability. In this section two methods will be discussed. The first is from [6], which discussed using the sign pattern of the underlying graph to determine its herdability.

In [5] it was shown that if a node was in a uni-signed column of the controllability matrix then it is herdable, where a uni-signed vector has all non-zero elements with the same sign. This was extended in [6] to show that if a node is in a balanced column of the controllability matrix, where a balanced vector has both positive and negative signs on the non-zero elements of the vector, and if the nodes with opposite sign are already known to be herdable then the node is herdable. Consider the following simple example system:

$$\dot{x} = \begin{bmatrix} \alpha_1 & 0 \\ \alpha_2 & 0 \end{bmatrix} x + \begin{bmatrix} \beta_1 \\ 0 \end{bmatrix} u,$$

for $\alpha_1, \alpha_2, \beta_1 > 0$. Then

$$\mathcal{C} = \begin{bmatrix} \beta_1 & \alpha_1 \beta_1 \\ 0 & -\alpha_2 \beta_1 \end{bmatrix}.$$

The first column of $\mathcal{C}$ is uni-signed, showing that $x_1$ is herdable. The second column is balanced, however as $x_1$ is herdable it can be used to show that $x_2$ is also herdable. Verifying herdability via sign pattern in this manner can be translated into Algorithm 1.

The results from the implementation of Algorithm 1 might be expected to be conservative, given that they only take

[1] A graph is structurally balanced if all semi-cycles (cycles when edge direction is ignored) have a positive sign [58].

**Algorithm 1** Checking Herdability based on Sign Pattern.

**Input:** $M \in \{\mathcal{C}, W_c, [A\ B]\}$
**Output:** $\mathbb{H}$ the set of herdable nodes
$\mathbb{H} \leftarrow \emptyset$
**for** $c$ in columns of $M$ **do**
  $P_c \leftarrow \{i | (M_{i,c} > 0)\}$
  $N_c \leftarrow= \{i | (M_{i,c} < 0)\}$
**end for**
**loop**
  **if** $P_c = \emptyset$ or $P_c \subseteq \mathbb{H}$ **then**
    $\mathbb{H} \leftarrow \mathbb{H} \cup N_c$
  **else if** if $N_c = \emptyset$ or $N_c \subseteq \mathbb{H}$ **then**
    $\mathbb{H} \leftarrow \mathbb{H} \cup P_c$
  **end if**
**end loop**

sign pattern into account. To validate the results based on sign pattern, the analysis is augmented with a computational method to determine the herdability of a system based on the controllability matrix $\mathcal{C}$. We define the cardinality herding problem as solving the following linear program:

$$\max_{u} \sum_{i=1}^{n} (\mathcal{C}u)_i \quad \text{s.t.} \quad \mathcal{C}u \leq 1_n. \tag{4}$$

Once the linear program is solved, the number of positive elements of the resultant vector $\mathcal{C}u$ is examined to determine how many states have been herded. This relatively simple optimization problem can be used to show that a large portion of a given network is herdable from one node.

A collection of complex networks from the literature are used in the analysis, though as mentioned previously the number of signed networks considered in the literature is quite small, resulting in far fewer networks analyzed in the general case. The networks are summarized in Table III. Each network has been checked for structural balance, based on the linear time algorithm of [59]. None of the networks examined are structurally balanced, i.e. the controllability matrix results hold for a specific weight combination and not for all networks that share the same sign pattern.

Each network referenced in Table III, has an associated signed adjacency matrix $\tilde{A}_s(\mathcal{G})$. It is assumed that the dynamics of the linear system which evolves over the network follows $A = \tilde{A}_s(\mathcal{G})$. Under these assumptions, all of the resultant linear systems are unstable. The signed networks are treated differently than the positive systems in Section IV-A as moving from a graph to consensus dynamics adds a negative self loop to each node, disrupting the underlying sign pattern of the network. A positive system remains a positive system with the addition of a negative self loop due to the definition of a Metzler matrix, [34], however adding a negative self loop may radically change the sign pattern of a linear system over a signed network. For example, if the underlying graph $\mathcal{G}$ was structurally balanced, then adding a self loop would make it no longer structurally balance. We leave an in-depth examination of the effect of these self loops to future work.

A consequence of the instability of the matrix $A$ is that the matrix product $A^m$ diverges numerically for some $m < n$. This implies that the controllability matrix can not be fully computed. However in the analysis that follows the controllability matrix is calculated until the resulting column has an element larger than a threshold (here $10^{10}$) allowing partial information on system herdability to be obtained.

Given the size of many of the networks, the herdability of the network was determined by selecting a random subset of 100 nodes and each node in that subset was considered in turn as the sole input. To consider the ability to herd from node $i$, it is assumed that $B = e_i$ and the herdability of the system is determined by either applying Algorithm 1 or the linear program in Eq. (4) to the controllability matrix. Table III shows the highest and lowest percentage of nodes that can be herded for the various methods.

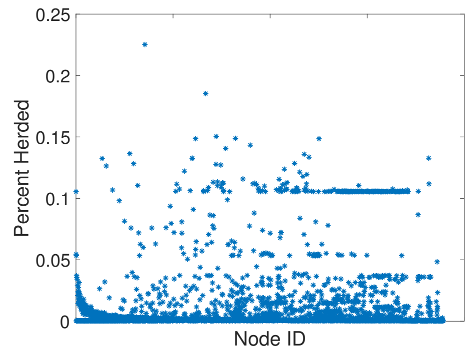



Fig. 4: Percent of system nodes herded on the Bitcoin Alpha network based on the sign pattern of the controllability matrix when taking each node as the sole input in turn.

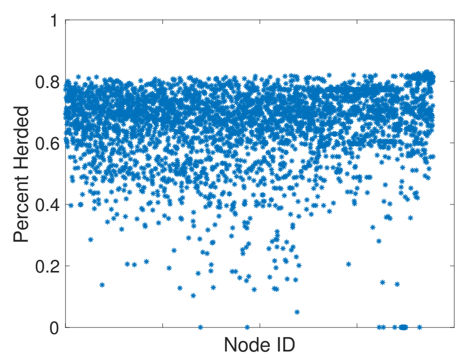



Fig. 5: Percent of system nodes herded on the Bitcoin Alpha network based on the cardinality herding analysis of the controllability matrix when taking each node as the sole input in turn.

Based on Table III, one can see that, even in the best case, the sign pattern of the controllability matrix predicts that only a small fraction of the network can be herded. However, solving the cardinality herding problem from Equation (4) shows that large fractions of the network can be herded, up to 83% from one node. As a sanity check for these results note

| Network Name | N | L | % Pos | $[A\ B]_h$ | $H_h$ | $H_l$ | $C_h$ | $C_l$ |
|---|---|---|---|---|---|---|---|---|
| Bitcoin Alpha [60] | 5,881 | 35,592 | 93 | 86.175 | 22.522 | 0.026 | 83.109 | 0.026 |
| Bitcoin OTC [60] | 3,783 | 24,186 | 89 | 79.221 | 17.650 | 0.017 | 76.365 | 0.017 |
| Slashdot 11/06/08 [61] | 77,357 | 516,575 | 77 | 50.385 | 0.019 | 0.001 | 38.517 | 0.001 |
| Slashdot 02/16/09 [61] | 81,871 | 545,671 | 77 | 49.562 | 0.024 | 0.001 | 36.699 | 0.001 |
| Slashdot 02/21/09 [61] | 82,144 | 549,202 | 77 | 50.464 | 0.052 | 0.001 | 35.789 | 0.001 |

TABLE III: Signed networks used to test system herdability: Each network has its name, number of nodes $N$, number of edges $L$, $\%$ Pos the fraction of positive edges, $[AB]_h$ the highest percentage of that can be herded based on the necessary condition on $[AB]$, $H_h$ the highest percentage of the network that can be herded based on the sign of $\mathcal{C}$, $H_l$ the lowest percentage of the network that can be herded based on the sign of $\mathcal{C}$, $C_h$ the highest percentage of the network that can be herded based on cardinality herding of $\mathcal{C}$, and $C_h$ the lowest percentage of the network that can be herded based on cardinality herding of $\mathcal{C}$.

that the highest percentage of herded nodes for each network is also lower than the upper bound based on the analysis of $[A\ B]$. It's also interesting to note that the gap between the upper bound and the maximum percentage herded based on cardinality herding is related to the fraction of positive edges in the network. This suggests that when there are more negative edges the system is harder to herd.

The results of Table III show the best and worst case for the two methods applied across a sample of input nodes. The question remains where the remaining nodes lie between the upper and lower bounds. In order to address this, every node in the Bitcoin Alpha network was taken as input and the percentage of nodes that can be herded was calculated for every node. The results are shown in Figure 4 and Figure 5. The sign pattern of the controllability matrix $\mathcal{C}$ suggests that certain nodes can herd significantly more nodes that others. However, as shown in Figure 5, based on cardinality herding problem the opposite is true, most nodes can herd $70-80\%$ of the network. This suggests that selecting a single node is sufficient to herd a large portion of the network. More work is required to extend these results to the multi input case, however it's likely that applying input to only a few nodes can ensure that the entire system is herdable.

## VI. Conclusion

This paper provides the first application of the notion of herdability to complex network data. Input selection for positive systems was shown to be possible in linear time. A novel centrality measure was introduced, which tends to select hubs to drive a system with minimum energy to a desired terminal set, even though hubs are not selected when considering the controllability of the system. It is shown that many centrality measures are not suitable for selecting herding nodes, especially in directed networks. In the general case, in which a linear system inherits a sign pattern from the underlying graph, it is shown that a large portion of signed networks can be herded from a single node. The results presented here are the beginning of a more nuanced understanding of the application of control theoretic ideas in complex networks. The notion of herdability examines more explicitly the existing assumptions about interacting with complex networks and in doing so helps bring new insight into the control theoretic characterization of complex networks.

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