A Viral Model of Product Adoption with Antagonistic Interactions

Sebastian F. Ruf, Philip E. Paré, Ji Liu, Carolyn L. Beck, and Tamer Başar

***Abstract*— In this paper, we extend a viral model for product adoption which takes into account how an agent's (or subpopulation's) opinion affects the decision to adopt a product or not. Here the coupled adoption opinion model considers the case where the opinion dynamic evolves over a signed network, which captures antagonistic interactions between agents. These signed networks capture a more realistic class of opinion behaviors and lead to a rich set of adoption behaviors for the coupled model. The equilibria of this model are characterized and some stability properties of these equilibria are discussed. Further behavior of the coupled model is studied via simulation.**

I. Introduction

Innovative technologies have repeatedly changed the world, from the printing press to the steam engine, to the transistor to the internet. The adoption of these technologies, as well as general adoption behavior, has been widely studied; particularly in sociology [4]–[6], [13], [30] and its sub-field, the diffusion of innovations [27], [33].

Models for the spread of a product often attempt to capture one of two mechanisms for its transmission. The first mechanism is contact with a product user. This interaction is often modeled as an epidemic, where the product spreads from one adopter of the product to a network neighbor who is not an adopter, i.e. the network neighbor subsequently adopts the product with some probability [3], [18], [34]. The second type of interaction is social reinforcement, commonly modeled by threshold models [13], [30], [33]. In threshold models, each agent has a threshold value, so that when the number of users in the neighborhood of the agent exceeds that value, the agent will adopt the product. Adoption that requires social reinforcement is often associated with adoption that involves some element of risk [4], which would include the products whose adoption is modeled in this paper.

In this paper, we use both mechanisms, following the approach of [28], [29], opting to explicitly model both the contact process, here an SIS epidemic model, and the social reinforcement, here an opinion dynamic. Both the SIS and opinion dynamic models have recently been of interest to the controls community [11], [23], [26], where the emphasis has been on analysis of their limiting behavior. For SIS epidemic models, this analysis has been based on network topology [9], [34], while for opinion dynamic models the analysis is often based on how different opinion dynamics change behavior. The study of opinion dynamic models has explored the consensus algorithms of Abelson and DeGroot [1], [7], the bounded confidence or Hegselmann-Krause model [16], and the signed consensus, or Altafini, model [2].

The work in this paper builds on past attempts to model these two adoption mechanisms simultaneously. Kalish extended a canonical epidemic model of product spread by Bass [3] by incorporating advertising in the dynamics. Similar to early SIS models such as [19], Bass' and Kalish's approaches assume a trivial underlying graph structure and model the system with only two differential equations, aggregating the population into groups. These simple models have been extended to allow a consumer's opinion about the quality of a product to affect their decision to purchase or adopt it. In [22], Martins et al. propose the Continuous Opinion Discrete Action (CODA) model which captures discrete product adoption with Bayesian opinion updates. However, in the CODA model, the Bayesian opinion update does not depend on any opinions, but only on the adoption actions of an agent's neighbors. In the model considered here, however, an agent's opinion updates based on its neighbors' opinions and its own adoption behavior and opinion.

As noted previously, the approach here follows that of [29] in which the authors proposed coupling several different opinion dynamic models with an SIS spread model to capture adoption behavior, concluding that the choice of opinion dynamic drives the outcome of the coupled system. Here we focus on a specific opinion dynamic, the signed consensus model, and seek to understand the effect of negative edges in the opinion graph on the spread of a product. These negative edges are typically conceptualized as interactions with an adversary in a social network [8], but can reflect any negative appraisal or opinion. Negative appraisal has been treated in the marking literature as oppositional brand loyalty [24], [25], in which an adopter defines themself not just by what they are adopting but also by what they are not adopting. For example, identifying as a Coke drinker means that one will not drink Pepsi. The signed consensus model has been considered in [29], but only in simulation.

In this paper, we characterize the equilibria of the viral model for production adoption coupled with the signed opinion dynamics and their stability properties. These equilibria are studied both analytically and numerically, and we show that antagonistic social relationships in the opinion dynamics have a strong effect on the adoption behavior.

Notation: Given a vector function of time $x(t)$, $\dot{x}(t)$ indicates the time-derivative. The time dependency is suppressed where it is clear from context, to simplify notation.

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The notation diag$(\cdot)$ refers to a diagonal matrix with the vector argument on the diagonal. The notation $\emptyset$ indicates the empty set. The N-dimensional vectors of zeros and ones are 0_N and 1_N, respectively. We also define $.5_N = .5 \times 1_N$ and $-.5_N = -.5 \times 1_N$. The matrix A formed from elements a_{ij} for $i \in \{1, 2, \ldots, N\}$ and $j \in \{1, 2, \ldots, N\}$ is denoted by $A = [a_{ij}]$. A matrix $A \in \mathbb{R}^{N\times N}$ is diagonally dominant if $\forall i \in \{1, 2, \ldots, N\}$ it holds that $|A_{ii}| \geq \sum_{j\neq i} |A_{ij}|$; strictly diagonally dominant means the inequality is strict.

II. Product Spread Model

We employ the modified spread dynamics proposed in [29] to incorporate the coupling between the "epidemic-like" spread of product adoption and the opinion exchange dynamics. The product adoption dynamics occur over a weighted, directed graph $\mathcal{G}_P$ of N subpopulations, or nodes. The opinion dynamics occur over a weighted, directed graph $\mathcal{G}_O$ with the same node set as $\mathcal{G}_P$, but whose edges may or may not coincide with $\mathcal{G}_P$. We say that agent j is a neighbor of node i in $\mathcal{G}_X$ if there is a directed edge from node j to node i in $\mathcal{G}_X$, where $X = P, O$, and denote the neighborhood set of node i as $\mathcal{N}_i^X$ for $X = P, O$.

Each node, or subpopulation, i has a proportion of agents that have adopted the product $x_i \in [0, 1]$. The subpopulation represented by node i also has an overall average opinion $o_i \in [0, 1]$, modeling how much the subpopulation values the product ($o_i = 0$ means the subpopulation is averse to the product, $o_i = 1$ means very receptive to the product). The product adoption dynamics for each node evolve as a function of time:

$$\dot{x}_i = -\delta_i x_i(1 - o_i) + (1 - x_i)o_i \left(\sum_{\mathcal{N}_i^P} \beta_{ij}x_j + \beta_{ii} \right) \quad (1)$$

where $\delta_i \geq 0$ is the product drop rate for subpopulation i, $\beta_{ij} \geq 0$ is the exogenous adoption rate representing the impact of node j on node i, and $\beta_{ii} \geq 0$ is the endogenous adoption rate . The parameters β_{ij} are also the weights on the product graph $\mathcal{G}_P$.

The two terms of the model, $\delta_i x_i(1 - o_i)$ and $(1 - x_i)o_i \left(\sum_{\mathcal{N}_i^P} \beta_{ij}x_j + \beta_{ii} \right)$, correspond roughly to a disadoption term and an adoption term, respectively. The disadoption term, $\delta_i(1 - o_i)x_i$, indicates that the rate of dropping the product is related to δ_i, just as in the healing term in the case of epidemic spread; however, here δ_i is modified by the average opinion o_i. A high opinion of the product will make the subpopulation less likely to disadopt. The adoption term, $(1 - x_i)o_i \left(\sum_{\mathcal{N}_i^P} \beta_{ij}x_j + \beta_{ii} \right)$, is modified in two respects. First, the rate of adoption is also modified by o_i, where a low o_i will decrease the rate of adoption of the product. Second, there is the term β_{ii} which describes the innovativeness of the population. If a population is innovative (large β_{ii}), it will adopt before its network neighbors do.

For the rest of the paper it is assumed that the initial conditions $x_i(0),\ o_i(0) \in [0, 1]\ \forall i$ and are known. As will be proven later, $x_i(0),\ o_i(0) \in [0, 1]\ \forall i$ implies that $x_i(t),\ o_i(t) \in [0, 1]\ \forall i,\ t \geq 0$. Hence, $x_i(t)$ and $o_i(t)$ are functions from $[0, \infty)$ to $[0, 1]$.

Assumption 1. For all $i \in \{1, 2, \ldots, N\}$, we have $\delta_i > 0$.

Assumption 2. The matrix $B = [\beta_{ij}]$ is non-negative and irreducible. Therefore $\mathcal{G}_P$ is strongly connected.

III. Signed Consensus Model

The signed consensus, or Altafini, model [2] is of the form

$$\dot{o}_i = \sum_{\mathcal{N}_i^O} |a_{ij}|(\text{sgn}(a_{ij})o_j - o_i), \quad (2)$$

where a_{ij} are real-valued weights, with negative weights for the neighbor nodes that the ith node distrusts. It is known that if the opinion graph is structurally balanced[1], then the model can reach a bipartite consensus, meaning that all the members of one group converge to a value and all the members of the other group converge to the negative of that value [2]. Alternatively, if the graph is structurally unbalanced, then the opinions converge to 0_N [2]. Due to the presence of negative opinions in the population, the following model was proposed in [29] to ensure that the full model is well defined:

$$\dot{x}_i = -\delta_i x_i(1 - o_i) + (1 - x_i)o_i \left(\sum_{\mathcal{N}_i^P} \beta_{ij}x_j + \beta_{ii} \right), \quad (3)$$

$$\dot{o}_i = \textstyle\sum_{\mathcal{N}_i^O} |a_{ij}|\big(\text{sgn}(a_{ij})(o_j - .5) - (o_i - .5)\big) + x_i - o_i, \quad (4)$$

where we assume that $\mathcal{G}_O$ is undirected. Define $o_i = \bar{o}_i + .5$ and assume that $\bar{o}_i(0) \in [-.5, .5]\ \forall i$. Then, (4) becomes

$$\dot{\bar{o}}_i = \sum_{\mathcal{N}_i^O} |a_{ij}|(\text{sgn}(a_{ij})\bar{o}_j - \bar{o}_i) + x_i - o_i, \quad (5)$$

which, without the $x_i - o_i$ term, reduces to the Altafini model. The term $x_i - o_i$ acts to ensure that a subpopulation reacts to adoption, which can be seen as removing cognitive dissonance from the agents in the subpopulation [10]. Cognitive dissonance occurs when there is a mismatch between an agent's behavior and the opinions that it holds. The Altafini dynamic, and the oppositional brand loyalty which it helps capture [24], [25], is particularly appropriate for products that involve a high degree of self-concept [32], i.e. products that relate to a person's self image. Those agents in the network which identify differently from each other will eventually clash, necessitating the Altafini model.

Note that when there are no negative edges this reduces to the Abelson model, the model that was explored in detail in [28], [29]. When negative edges are present and the graph is structurally balanced, the system can converge to a split equilibrium, that is, one where some subpopulations have adopted and some subpopulations have disadopted. This is illustrated via simulation in Section V.

IV. Analysis

We first show that the model in (3) and (5) remains in a compact set. We then provide a series of propositions that characterize some of the equilibria of the model.

Lemma 1. For the model in (5), if $x(0) \in [0, 1]^N$ and $\bar{o}(0) \in [-.5, .5]^N$, then $x(t) \in [0, 1]^N$ and $\bar{o}(t) \in [-.5, .5]^N,\ \forall t \geq 0$.

[1]A signed graph is *structurally balanced* if it has a bipartition of the nodes V_1, V_2, i.e., $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$, such that $a_{ij} \leq 0$, $\forall v_i \in V_p$, $v_j \in V_q$ where $p, q \in \{1, 2\}$, $p \neq q$; otherwise, $a_{ij} \geq 0$ [15].

Proof. First consider the opinion dynamic. Equation (2) can be rewritten as

$$\dot{o} = -\mathcal{L}_O o, \tag{6}$$

where $\mathcal{L}_O$ is the Laplacian and each element of the spectrum of $-\mathcal{L}_O$ is non-positive, i.e. $\forall \lambda \in \sigma(-\mathcal{L}_O),\ \lambda \leq 0$ [2] (undirected $\mathcal{G}_O$ implies an all real spectrum). This implies that the system is diminishing or oscillating, that is, no state is increasing in magnitude. We can rewrite (5) as

$$\dot{\bar{o}} = (-\mathcal{L}_O - I)\bar{o} + x - .5_N.$$

To consider the effect of $x(t)$ on the opinion dynamic, suppose that $x(t) \in [0,1]^N$. It follows that $x - .5_N \in [-.5, .5]^N$, and thus any positive effect $x - .5_N$ has on the system will be dampened by the $-\bar{o}$ term. Therefore if $\bar{o}(0) \in [-.5,.5]^N$ and $x(t) \in [0,1]^N$, then $\bar{o}_i(t) \in [-.5,.5],\ \forall t \geq 0$.

Next consider the adoption dynamic and suppose that $\bar{o}(t) \in [-.5,.5]^N$. Then, we have $o \in [0,1]^N$. So, by (3), if for some fixed t^*, $x_i(t^*) = 1$, then $\dot{x}_i(t^*) = -\delta_i(1 - o_i) \leq 0$. Likewise, if for some t_*, $x_i(t_*) = 0$, then $\dot{x}_i(t_*) = o_i\left(\sum_{\mathcal{N}_i^P} \beta_{ij}x_j + \beta_{ii}\right) \geq 0$. Therefore, since each $x_i(t)$ is a continuous function, if $x(0) \in [0,1]^N$ and $o(t) \in [0,1]^N$ we have $x(t) \in [0,1]^N,\ \forall t \geq 0$. Since $x(0) \in [0,1]^N$ and $\bar{o}(0) \in [-.5,.5]^N$, from the preceding discussion, $x(t) \in [0,1]^N$ and $\bar{o}(t) \in [-.5,.5]^N,\ \forall t \geq 0$. ☐

Proposition 1. Under Assumption 1, if there exists an edge in $\mathcal{G}_O$ such that $\text{sgn}(a_{ij}) = -1$, then there can be no equilibrium with $o_i = 0,\ \forall i$. If there are no negative edges, then $z^* = 0_{2N}$ is an equilibrium of the system.

Proof. If there are no negative edges, it is easy to see from (3) and (4) that $z^* = 0_{2N}$ is an equilibrium point for the system. Now consider the case when there exists an edge such that $\text{sgn}(a_{ij}) = -1$. Suppose that, to the contrary, there exists an equilibrium with $o_i = 0,\ \forall i$, which implies that $\bar{o}_i = -.5,\ \forall i$. From (3), it follows that $x_i^* = 0,\ \forall i$. Then, (4) reduces to the continuous-time Altafini model. If the signed graph is structurally balanced with at least one negative edge, the signed Laplacian matrix has an eigenvalue at zero, whose corresponding eigenvector cannot be 1_N [2]. Thus, $-.5_N$ cannot be an eigenvector of the signed Laplacian matrix for the zero eigenvalue, which contradicts the hypothesis that there exists an equilibrium with $o_i = 0,\ \forall i$. If the signed graph is structurally unbalanced, the signed Laplacian matrix does not have an eigenvalue at zero [2]. Thus, $\mathcal{L}_O\bar{o} \neq 0_N$, which also contradicts the hypothesis that there exists an equilibrium with $o_i = 0,\ \forall i$. ☐

Proposition 2. Under Assumption 1, if there exists an edge such that $\text{sgn}(a_{ij}) = -1$, then there can be no equilibrium with $o_i = 1,\ \forall i$. If there are no negative edges, $z^* = 1_{2N}$ is an equilibrium of the system.

Proof. The proposition can be proved using arguments similar to those in the proof of Proposition 1. ☐

Proposition 3. The equilibria in x_i of the system are of the form

$$x_i^* = \frac{o_i^*\left(\sum_{\mathcal{N}_i^P} \beta_{ij}x_j^* + \beta_{ii}\right)}{\delta_i(1-o_i^*) + o_i^*\left(\sum_{\mathcal{N}_i^P} \beta_{ij}x_j^* + \beta_{ii}\right)}.$$

Proof. Consider the x_i dynamic at equilibrium:

$$0 = -\delta_i x_i^*(1-o_i^*) + (1-x_i^*)o_i^*\left(\sum_{\mathcal{N}_i^P} \beta_{ij}x_j^* + \beta_{ii}\right)$$

$$\delta_i x_i^*(1-o_i^*) = (1-x_i^*)o_i^*\left(\sum_{\mathcal{N}_i^P} \beta_{ij}x_j^* + \beta_{ii}\right)$$

$$\left(\delta_i(1-o_i^*) + o_i^*\left(\sum_{\mathcal{N}_i^P} \beta_{ij}x_j^* + \beta_{ii}\right)\right)x_i^* = o_i^*\left(\sum_{\mathcal{N}_i^P} \beta_{ij}x_j^* + \beta_{ii}\right)$$

$$x_i^* = \frac{o_i^*\left(\sum_{\mathcal{N}_i^P} \beta_{ij}x_j^* + \beta_{ii}\right)}{\delta_i(1-o_i^*) + o_i^*\left(\sum_{\mathcal{N}_i^P} \beta_{ij}x_j^* + \beta_{ii}\right)}.$$

☐

Proposition 4. If $\beta_{ii} = 0,\ \forall i$, then there is an equilibrium point of the system such that $x^* = 0_N$.

Proof. From Proposition 1, this holds if there are no negative edges in the system. Consider the case where there is at least one negative edge in the opinion graph $\mathcal{G}_O$ and denote the equilibrium opinion as $o^* \neq 0_N$. Consider the adoption dynamic at $x = 0_N$:

$$\begin{aligned}\dot{x}_i &= -\delta_i x_i(1-o_i^*) + (1-x_i)o_i^*\left(\sum_{\mathcal{N}_i^P} \beta_{ij}x_j + \beta_{ii}\right)\\ &= -\delta_i(0)(1-o_i^*) + (1-0)o_i^*\left(\sum_{\mathcal{N}_i^P} \beta_{ij}0 + \beta_{ii}\right)\\ &= o_i^*\beta_{ii}.\end{aligned}$$

Then if $\beta_{ii} = 0$, $\dot{x}_i = 0$. If this holds $\forall i$, then $x = 0_N$ is an equilibrium in the adoption dynamic. ☐

Proposition 5. If $\delta_i > \sum_{\mathcal{N}_i^P} \beta_{ij} + \beta_{ii}$, then at equilibrium $x_i^* < o_i^*$. If $\delta_i < \beta_{ii}$, then at equilibrium $o_i^* < x_i^*$. Given x^*, if $\delta_i > \sum_{\mathcal{N}_i^P} \beta_{ij}x_j^* + \beta_{ii}$, then at equilibrium $x_i^* < o_i^*$, or if $\delta_i < \sum_{\mathcal{N}_i^P} \beta_{ij}x_j^* + \beta_{ii}$, then at equilibrium $o_i^* < x_i^*$.

Proof. By assumption, $\delta_i > \sum_{\mathcal{N}_i^P} \beta_{ij} + \beta_{ii}$, and since $x_j \in [0,1]\ \forall j$, we have $\delta_i > \sum_{\mathcal{N}_i^P} \beta_{ij}x_j^* + \beta_{ii}$, which implies that

$$\delta_i(1-o_i^*) + o_i^*\left(\sum_{\mathcal{N}_i^P} \beta_{ij}x_j^* + \beta_{ii}\right) > \left(\sum_{\mathcal{N}_i^P} \beta_{ij}x_j^* + \beta_{ii}\right).$$

Therefore,

$$x_i^* = \frac{o_i^*\left(\sum_{\mathcal{N}_i^P}\beta_{ij}x_j^* + \beta_{ii}\right)}{\delta_i(1-o_i^*) + o_i^*\left(\sum_{\mathcal{N}_i^P}\beta_{ij}x_j^* + \beta_{ii}\right)} < \frac{o_i^*\left(\sum_{\mathcal{N}_i^P}\beta_{ij}x_j^* + \beta_{ii}\right)}{\left(\sum_{\mathcal{N}_i^P}\beta_{ij}x_j^* + \beta_{ii}\right)} = o_i^*. \quad (7)$$

If $\delta_i < \beta_{ii}$ then by similar logic $x_i^* > o_i^*$. Given x^*, the remaining propositions follow similarly. □

We next describe the behavior of the system when there is strong disadoption, i.e. $\delta_i > \sum_{\mathcal{N}_i^P}\beta_{ij} + \beta_{ii}$. This is captured by the following main result of this section which, under certain conditions, characterizes the unique equilibrium of the model, with local exponential stability.

Theorem 1. Suppose that $\delta_i > \sum_{\mathcal{N}_i^P}\beta_{ij} + \beta_{ii}$, $\beta_{ii} = 0\ \forall i$, and $o^* < .5_N$. Then, under Assumption 2, the model has a unique locally exponentially stable equilibrium, $x^* = 0_N$.

Proof. From (3) and with $\beta_{ii} = 0$, the dynamics of x can be regarded as an SIS model with healing rate $\delta_i(1-o_i)$ and infection rates $o_i\beta_{ij}$. For a given o, it is known that if $s(-D(I-O)+OB) \le 0$, x has a unique equilibrium at 0_N, where $s(M)$ denotes the largest real part among all eigenvalues of square matrix M (see Propositions 3 and 4 in [20]), $D = \text{diag}(\delta_1,\dots,\delta_N)$, $O = \text{diag}(o_1,\dots,o_N)$, and $B = [\beta_{ij}]$. Since $\delta_i > \sum_{\mathcal{N}_i^P}\beta_{ij}$ and $o^* < .5_N$, it follows that $(1-o_i^*)\delta_i > o_i^*\sum_{\mathcal{N}_i^P}\beta_{ij}$, $\forall i$, which implies that $s(-D(I-O)+OB) < 0$. Thus, $x^* = 0_N$, which is unique. Since $x^* = 0_N$, from (4), $(\mathcal{L}_O + I)o^* = \mathcal{L}_O .5_N$.

Since $(\mathcal{L}_O + I)$ is invertible, no matter whether $\mathcal{G}_O$ is structurally balanced or not, o^* is unique. Thus, the model has a unique equilibrium.

To show stability of this equilibrium, consider the Jacobian at equilibrium, which is of the form:

$$J(z^*) = \left[\begin{array}{c|c} J_1 & 0_{N\times N} \\ \hline I & -(\mathcal{L}_O + I)\end{array}\right], \quad (8)$$

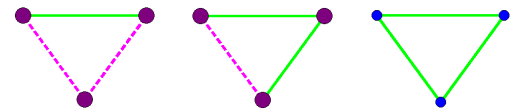
where $J_1 \in \mathbb{R}^{N\times N}$ satisfies for each $i \in \{1,2,\dots,N\}$ that:

$$J_1(z^*)_{ii} = -\delta_i(1-o_i^*)$$
$$J_1(z^*)_{ij} = \begin{cases} o_i\beta_{ij} & \text{if } j \in \mathcal{N}_i^P \\ 0 & \text{if } j \notin \mathcal{N}_i^P \cup \{i\}. \end{cases} \quad (9)$$

To determine the stability of the equilibrium we now show that the Jacobian is diagonally dominant and non-singular. The second N rows of $J(z^*)$ are diagonally dominant by definition. Consider row i with $i \in \{1,2,\dots,N\}$ of $J(z^*)$:

$$\begin{aligned} |J(z^*)_{ii}| &= |J_1(z^*)_{ii}| \\ &= |-\delta_i(1-o_i^*)| = \delta_i(1-o_i^*) \\ &> \sum_{\mathcal{N}_i^P}\beta_{ij}o_i = \sum_{j\ne i}|J(z^*)_{ij}| \end{aligned} \quad (10)$$

Therefore the first N rows of $J(z^*)$ are strictly diagonally dominant. As the diagonal elements of $J(z^*)$ are negative and $J(z^*)$ is diagonally dominant, the Gershgorin disc theorem shows that the eigenvalues of $J(z^*)$ are non-positive [17]. The structure of $J(z^*)$ is such that for each row i where $J(z^*)$ is not strictly diagonally dominant, i.e. $|J(z^*)_{ii}| \ge \sum_{j\ne i}|J(z^*)_{ij}|$, the element $J(z^*)_{i,i-N} = 1 > 0, \forall i \in \{N+1,\dots,2N\}$. This implies $J(z^*)$ has no eigenvalues at zero [31]. So the eigenvalues of $J(z^*)$ are negative and the equilibrium is locally exponentially stable. □



(a) Using A_1 in (11) for $\mathcal{G}_O$ (b) Using A_2 in (12) for $\mathcal{G}_O$ (c) Using A_3 in (13) for $\mathcal{G}_O$

Fig. 1: Equilibria using the model in (3)-(4), $\mathcal{G}_O$ with adjacency matrices in (11)-(13), and $\mathcal{G}_P$ given by (14).

Based on the simulations in Section V, we conjecture that the condition $o^* < .5_N$ will hold under the following:

Conjecture 1. If at equilibrium $x^* < o^*$, then $o^* < .5_N$.

If true, this conjecture would imply that if $\delta_i > \sum_{\mathcal{N}_i^P}\beta_{ij} + \beta_{ii}$ then $o^* < .5_N$ by Proposition 5 and if $\beta_{ii} = 0$, $\forall i$ then Theorem 1 could be used to show stability.

V. Simulations

In this section, the discussion moves from the theoretical characterization of the equilibria of the coupled adoption opinion model to various simulations that explore the behavior of the model. For the figures, green and dashed magenta lines indicate positive and negative edges, respectively, in the opinion graph $\mathcal{G}_O$. For the nodes, blue (b) indicates $x_i = 0$, red (r) indicates $x_i = 1$, and the color of node i shown is determined by $x_i r + (1-x_i)b$. The diameter of each node i is scaled by o_i.

As mentioned in [29] this model can exhibit bipartite consensus behavior, which will be discussed further using three examples from [2]. We use the matrices from these examples as the corresponding adjacency matrices for the opinion graph $\mathcal{G}_O$:

$$A_1 = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & -4 \\ -2 & -4 & 0 \end{bmatrix}, \quad (11)$$

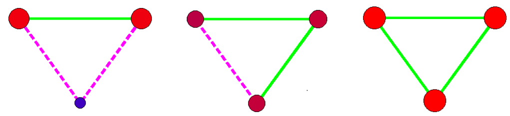
$$A_2 = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 4 \\ -2 & 4 & 0 \end{bmatrix}, \text{ and} \quad (12)$$

$$A_3 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}. \quad (13)$$

We set the product spread matrix $B = [\beta_{ij}]$ to

$$B = \beta\hat{A}_1 = \beta\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad (14)$$

with $\beta = 0.5 = \delta_i\ \forall i$. Here, $\beta_{ii} = 0$, $\forall i$. The equilibria for these systems are depicted in Figure 1. The final opinions are given by $\bar{o}_1^* = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, $\bar{o}_2^* = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, $\bar{o}_3^* = \begin{bmatrix} -0.5 & -0.5 & -0.5 \end{bmatrix}$, and the final product adoption states are $x_1^* = \begin{bmatrix} 0.5 & 0.5 & 0.5 \end{bmatrix}$, $x_2^* = \begin{bmatrix} 0.5 & 0.5 & 0.5 \end{bmatrix}$, $x_3^* = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, where we abuse notation using the subscript i to indicate the use of A_i for the opinion graph $\mathcal{G}^O$. These equilibria appear to be unique, independent of the initial condition (as long as $\bar{o}(0) \in [-.5, .5]^N$ and $x(0) \in [0,1]^N \setminus \{0_N\}$).



(a) Using A_1 in (11) for $\mathcal{G}_O$ (b) Using A_2 in (12) for $\mathcal{G}_O$ (c) Using A_3 in (13) for $\mathcal{G}_O$

Fig. 2: Equilibria using the model in (3)-(4), the adjacency matrices in (11)-(13), and $B = \beta(\hat{A}_1 + I)$.

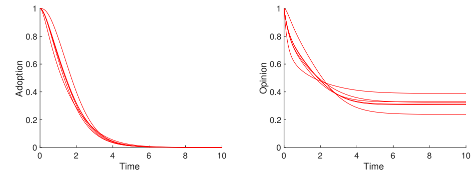
If we allow $\beta_{ii} = 0.5,\ \forall i$, that is, $B = \beta(\hat{A}_1 + I)$, then the behaviors change. The equilibria for these systems are depicted in Figure 2. The final opinions are given by $\bar{o}_1^* = \begin{bmatrix}0.3887 & 0.3811 & -0.3647\end{bmatrix}$, $\bar{o}_2^* = \begin{bmatrix}0.0390 & 0.1171 & 0.0944\end{bmatrix}$, $\bar{o}_3^* = \begin{bmatrix}0.5 & 0.5 & 0.5\end{bmatrix}$, and the final product adoption states are $x_1^* = \begin{bmatrix}0.9446 & 0.9407 & 0.2508\end{bmatrix}$, $x_2^* = \begin{bmatrix}0.7275 & 0.7864 & 0.7699\end{bmatrix}$, $x_3^* = \begin{bmatrix}1.00 & 1.00 & 1.00\end{bmatrix}$ The equilibria appear once again to be unique, independent of initial condition (as long as $\bar{o}(0) \in [-.5, .5]^N$ and $x(0) \in [0,1]^N \setminus \{0_N\}$). Note that for the structurally balanced opinion graph in Figure 2a the system converges to a bipartite consensus. However the system depicted in Figure 1a, with the same structurally balanced opinion graph does not converge to a bipartite consensus. This discrepancy illustrates the importance of the β_{ii} terms, which differentiate the two systems. Also note that, as expected, the systems in Figures 1c and 2c behave the same as the Abelson model studied in [29].

We extend the simulations to a 6-node graph to explore the behaviors with dominating healing rates and infection rates on nontrivial graphs. The following figures show time series data to help characterize the complex behavior of the coupled adoption opinion system. The 6-node graph can be described using three matrices for $\mathcal{G}_O$: $A_4 = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$,

$$A_5 = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \quad \hat{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

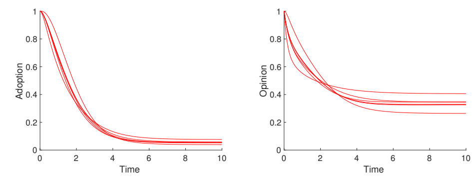
As shown by Theorem 1, if $\delta_i > \sum_{\mathcal{N}_i^P} \beta_{ij}, \forall i,\ \beta_{ii} = 0, \forall i$ and there is a negative edge in the graph, then there is a unique, locally stable equilibrium. Figure 3 considers the case where the opinion graph has adjacency matrix A_4, $B = .2\hat{A}_2$ and $\delta_i = 2, \forall i$ and shows that this equilibrium seems to be asymptotically stable on $[0,1]^{2N}$.

Figure 4 shows the system behavior where $\mathcal{G}_O$ has adjacency matrix A_4, $B = .2(\hat{A}_2 + I)$ and $\delta_i = 2, \forall i$. Figure 4 suggests that there is a unique, asymptotically stable equilibrium point that satisfies $0 < x_i^* < o_i^* < .5,\ \forall i$ under such conditions. The ordering of the adoption and opinion at equilibrium does not seem to rely on the structural balance of the underlying opinion network; this is shown in Figure 5, which considers the same adoption parameters as Figure 4



(a) Adoption (b) Opinion

Fig. 3: Adoption and Opinion for a structurally balanced 6-node network with $\delta_i > \sum_{\mathcal{N}_i^P} \beta_{ij} + \beta_{ii},\ \forall i$ and $\beta_{ii} = 0,\ \forall i$.



(a) Adoption (b) Opinion

Fig. 4: Adoption and Opinion for a structurally balanced 6-node network with $\delta_i > \sum_{\mathcal{N}_i^P} \beta_{ij} + \beta_{ii},\ \forall i$ and $\beta_{ii} \neq 0,\ \forall i$.

however the opinion graph has adjacency matrix A_5, which is not structurally balanced, instead of A_4, which is structurally balanced. Although the equilibrium point has changed, it once again satisfies the ordering $0 < x_i^* < o_i^* < .5,\ \forall i$. A comparison of the structures and equilibria of Figure 4 and Figure 5 are shown in Figure 6. The results of the simulations lead to the following conjecture:

Conjecture 2. If $\delta_i > \sum_{\mathcal{N}_i^P} \beta_{ij} + \beta_{ii},\ \forall i,\ \beta_{ii} > 0,\ \forall i,$ and there is a negative edge in the opinion graph, then there is a unique, endemic equilibrium for the system such that $.5 > o_i > x_i > 0,\ \forall i$ which is stable on $[0,1]^{2N}$.

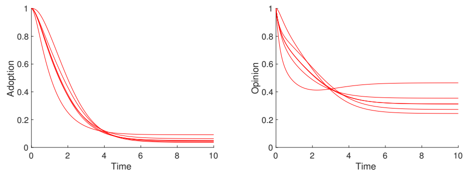
Note that the stability on $[0,1]^{2N}$ is possible due to the negative edge and the fact that $\beta_{ii} > 0,\ \forall i$ which excludes 1_{2N} and 0_{2N} as equilibria by Proposition 2. Conjecture 2 suggests that negative edges in a network have a powerful effect on the disadoption behavior of the system. If there is a single negative tie, then even if the innovation is as undesirable as possible, there will still be those that use the product. This might represent cases where self image, expressed through opposition between communities, is sufficient to encourage a fraction of the network to ignore product quality.

We also consider the high adoption rate scenario, i.e. $\beta_{ii} > \delta_i, \forall i$. Figure 7 shows the case when the opinion graph has an adjacency matrix A_4, $B = .2(\hat{A}_2 + I)$ and $\delta_i = .1, \forall i$. Under these conditions, there appears to be an equilibrium which satisfies $1 > x_i^* > o_i^* > .5$ at equilibrium.

Conjecture 3. If $0 < \delta_i < \beta_{ii}$ and there is a negative edge in the opinion graph, then there is a unique, stable endemic equilibrium for the system such that $1 > x_i^* > o_i^* > .5$ which is stable on $[0,1]^{2N}$.

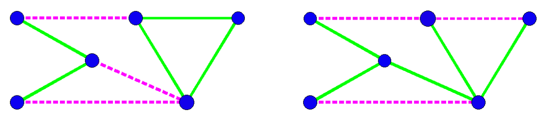
Again, a negative edge has a strong effect on the behavior of the model. If one negative edge exists, even if the product is as viral as possible not everyone will adopt.

One way to think about the negative edges and their impact is through political polarization, where a negative edge rep-



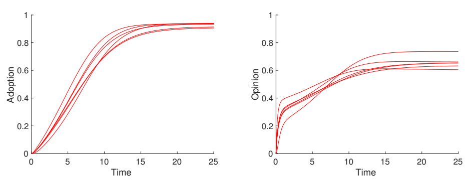
(a) Adoption (b) Opinion

Fig. 5: Adoption and Opinion for a 6-node network that is not structurally balanced with $\delta_i > \sum_{\mathcal{N}_i^P} \beta_{ij} + \beta_{ii},\ \forall i$ and $\beta_{ii} \neq 0,\ \forall i$.



(a) Equilibrium of Fig. 4 (b) Equilibrium of Fig. 5

Fig. 6: Opinion graph structures and equilibria of the coupled adoption opinion model for two sets of parameters.



(a) Adoption (b) Opinion

Fig. 7: Adoption and Opinion for a structurally balanced 6-node network with $0 < \delta_i < \beta_{ii},\ \forall i$.

resents two communities that have become polarized against each other. Recent studies have suggested that polarization has an impact on the adoption of ideas about climate change [14] and can effect brand value [21]. Conjectures 2 and 3 suggest that the model presented here has the potential to help understand adoption behavior under polarization, which is particularly pressing in light of recent events such as the Nike advertising campaign featuring Colin Kaepernick [12].

VI. Conclusion

We have explored a product adoption model which allows antagonistic interaction in the opinion dynamics. We have shown that the model is well posed, and have provided preliminary analyses. The behavior of the model have been further explored via simulation, resulting in several conjectures, whose proofs will be subjects of future work.

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