Dynamics of Opinion-Dependent Product Spread

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Abstract— **In this paper we propose a novel model of product spread that combines models which capture the behavior of epidemic-like product spread among agents in a network with different opinion models, allowing a greater range of product spread behaviors to be modeled. In the proposed opinion-dependent product spread models, product adoptions over a network are affected by the agents' opinions of the product. These opinions evolve via a separate set of network-dependent opinion dynamics, which are also dependent on the product adoption of the agent. The behavior of the product spreading is explored under the influence of several different models of opinion dynamics. We provide analysis of the local equilibria of the coupled models and compare the behavior of the coupled product-opinion dynamics via simulation. These simulations illustrate that the opinion dynamics drive the outcome of the system, allowing different types of product adoption to be modeled through different choices of opinion dynamics and strengthening the ability of epidemic spread techniques to model product adoption.**

I. Introduction

A consumer's choice to adopt a product results from a complex interplay between the behavior of the consumer, the behavior of its network neighbors, and the consumer's opinion towards the product [1]. More accurately describing this process requires models that capture both the spread of a product and the spread of an opinion. The SIS (susceptible-infected-susceptible) epidemic model abstractly describes a spreading process of a single disease, product, norm, or idea [2], [3]. Both SIS epidemic models and opinion dynamics have been extensively studied in isolation but have yet to be examined together. We investigate new coevolutionary dynamics that arise when they are coupled, giving rise to a model capturing the effect consumers' opinions have on the epidemic-like spread of a single product, norm, or idea. This approach allows for a more nuanced description of epidemic product spread, providing avenues to address the issue of a lack of diversity in outcome, which has been raised for using epidemic models for product spread [3].

Networked SIS models are well-understood mathematically and computationally [4], [5]. In such models, an individual is either susceptible to infection (has not yet adopted the product or idea) or is infected (has adopted). Susceptible individuals can become infected from each of their infected neighbors in the network at a rate β, and infected individuals can heal to become susceptible again at a rate δ. In the setting of epidemic awareness and behavior change, SIS models with dynamically scaled β parameters have been studied [6]–[8]. In these works, the coevolution between epidemic and behavior can only hinder the spread of infection. The combined product-opinion model proposed herein essentially modifies β and δ based on the agent's opinion towards the product, and can either hinder or promote the spread of the product.

Opinion dynamics have been of interest in sociology since the canonical models of Abelson and DeGroot [9], [10], and have since become of interest to the controls community [11]. These models have been extended in search of outcomes that do not reach "universal ultimate agreement" by the Hegselmann-Krause and Altafini models [12], [13]. The Hegselmann-Krause model allows agents to ignore others whose opinions are sufficiently different, while the Altafini model allows agents to have negative interactions with network neighbors. Hence, even the relatively simple Abelson and DeGroot models can be modified to capture several ways that opinions spread through a population.

There have been few works that study the interplay between product adoption and opinion dynamics, i.e. allowing a consumer's opinion about the quality or value of a product affect his/her decision to purchase or adopt it. These opinions change dynamically because they are influenced by the opinions and decisions of their network neighbors. Kalish proposed a coupled adoption and awareness model that includes advertising [14]. Similar to early SIS models, this model assumes full connectivity of the graph and models the system with only two differential equations, aggregating the population into one group. The Continuous Opinion Discrete Action (CODA) model provides a model of discrete product adoption with Bayesian opinion updates [15]. However the Bayesian opinion update only depends on the adoption actions of network neighbors, not their opinions.

In this paper, we study dynamics that couple a mean-field SIS epidemic ODE model with various continuous-time opinion dynamics over a network of agents. Depending on how opinions propagate, we observe qualitatively different dynamics. Hence, the choice of opinion dynamic drives the outcome of the coupled system, bringing the model closer to the reality that a consumer's opinion of a product drives their propensity to adopt or avoid said product. For example, new technologies or ideologies can become the new collective standard because they introduce a much-needed change in society. On the other hand, they can be rejected in favor of the status quo if the population is not receptive to change. There

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can also be coexistence among several competing products or ideas, e.g. a consumer's preferred brand of smart phones or an individual's political affiliation. Our opinion-dependent product spread models exhibit behavior that reflects all of these scenarios, by looking at two distinct models of opinion dynamics. The first is the Abelson consensus model, which reaches universal ultimate agreement. For the second, we propose a novel threshold-based opinion dynamic model which exhibits polarized opinions in clusters of the network.

The paper is organized as follows: The product spread model is introduced in Section II, which is a mean-field SIS epidemic ODE in continuous time that is modified via coupling with opinion exchange dynamics among the nodes in the network. In Section III, we couple the product adoption model with a Laplacian-based consensus model, while Section IV explores coupling with a threshold-based opinion dynamic. Simulation-based comparisons between the two opinion models are given in Section V. Finally, we give concluding remarks in Section VI.

A. Notation

Given a vector $x(t)$, $\dot{x}(t)$ indicates the time-derivative. The time dependency is dropped where it is obvious, to simplify notation. The notation $\frac{\partial f}{\partial x}$ indicates the partial derivative of f with respect to x. Given a vector $x \in \mathbb{R}^N$, the transpose is denoted by x^T. The notation diag$(\cdot)$ refers to a diagonal matrix with the vector argument on the diagonal. The notation $\emptyset$ indicates the empty set. The N-dimensional vectors of zeros and ones are 0_N and 1_N, respectively.

II. PRODUCT SPREAD MODEL

We modify the standard SIS epidemic ODE dynamics to incorporate the coupling between the "epidemic-like" spread of product adoption and the opinion exchange dynamics. The product adoption dynamics occur over a weighted, directed network $\mathcal{G}_P$ of N agents, or nodes. The opinion dynamics occur over a weighted digraph $\mathcal{G}_O$ with the same node set as $\mathcal{G}_P$, but whose edges may or may not coincide with $\mathcal{G}_P$. We denote the neighborhood set of agent i as $\mathcal{N}_i^X$ for $X = P, O$.

Each node i has an adoption probability $x_i \in [0,1]$ for the product, which represents how likely the consumer is to adopt the product ($x_i = 0$ means the consumer has not adopted, $x_i = 1$ means the consumer has). The consumer represented by node i also has an opinion $o_i \in [0,1]$, modeling how much the consumer values the product ($o_i = 0$ means very averse to the product, $o_i = 1$ means very receptive to the product). The product adoption dynamics for each node evolve as a function of time:

$$\begin{aligned} \dot{x}_i &= f_i(x,o) \\ &\equiv -\delta_i x_i(1-o_i) + (1-x_i)o_i\left(\sum_{\mathcal{N}_i^P}\beta_{ij}x_j + \beta_{ii}\right) \end{aligned} \quad (1)$$

where $\delta_i \geq 0$ is the product drop rate for agent i, $\beta_{ij} \geq 0$ the exogenous adoption rate, and $\beta_{ii} \geq 0$ the endogenous adoption rate. The parameters β_{ij} are the weights on the product graph. It is assumed the initial conditions $x_i(0), o_i(0) \in [0,1]$ $\forall i$ are known. As will be shown later, $x_i(0), o_i(0) \in [0,1]$ $\forall i$ implies $x_i(t), o_i(t) \in [0,1]$ $\forall i, t \geq 0$. Hence, $x_i(t)$ and $o_i(t)$ are functions from $[0,\infty)$ to $[0,1]$.

In the subsequent sections we will explore two different opinion dynamic models and their effects on the spread of the product. When convenient, we denote the aggregate $2N$-state vector by $z = [x^T, o^T]^T$.

For the model in (1), each x_i represents a probability of adoption and each o_i is a scaled opinion. As such the proposed model is only meaningful for $x_i, o_i \in [0,1]$. To this end we first establish well-posedness of the product model.

Lemma 1: For the model in (1), if $x(0) \in [0,1]^N$ and $o(t) \in [0,1]^N$ for all $t \geq 0$, then $x_i(t) \in [0,1]$ for all $t \geq 0$.

Proof: Assume $o(t) \in [0,1]$ for all $t \geq 0$.
If $x_i(0) = 0$ and $x_j(0) \in [0,1]$ for all $j \neq i$, then by (1), $\dot{x}_i(0) \geq 0$, driving $x_i(t) \geq 0$ for $t > 0$, since $\beta_{ij} \geq 0$.
If $x_i(0) = 1$ and $x_j(0) \in [0,1]$ for all $j \neq i$, then by (1), $\dot{x}_i(0) = -\delta_i x_i(1-o_i) \leq 0$, driving $x_i(t) \leq 1$ for $t > 0$, since $\delta_i \geq 0$.

Since there exists a derivative by (1), $x_i(t)$ is continuous. Therefore since we assume $x_i(0) \in [0,1]$ for all i, and have shown that for t such that $x_i(t) = 1$, $\dot{x}_i(t) \leq 0$ and for t such that $x_i(t) = 0$, $\dot{x}_i(t) \geq 0$, we have $x_i(t) \in [0,1]$ for all $t \geq 0$. ■

Having shown the well-posedness of the product model, we now discuss properties of the product spread model by considering the partial derivatives of the function in (1). Note

$$\frac{\partial f_i}{\partial x_i} = -\delta_i(1-o_i) - o_i\left(\sum_{\mathcal{N}_i^P}\beta_{ij}x_j + \beta_{ii}\right), \quad (2)$$

which is always negative under the assumptions of Lemma 1 since $\beta_{ij}, \delta_i \geq 0$. The other set of partial derivatives with respect to x is

$$\frac{\partial f_i}{\partial x_j} = \begin{cases} (1-x_i)o_i\beta_{ij} & \text{if } j \in \mathcal{N}_i^P, j \neq i \\ 0 & \text{if } j \notin \mathcal{N}_i^P \cup \{i\}, \end{cases}$$

which is always non-negative under the assumptions of Lemma 1 and since $\beta_{ij} \geq 0$. We also have

$$\frac{\partial f_i}{\partial o_i} = \delta_i x_i + (1-x_i)\left(\sum_{\mathcal{N}_i^P}\beta_{ij}x_j + \beta_{ii}\right) \quad (3)$$

which is always non-negative under the assumptions of Lemma 1 and since $\beta_{ij}, \delta_i \geq 0$. Finally,

$$\frac{\partial f_i}{\partial o_j} = 0 \ \forall j \neq i. \quad (4)$$

As in the classic SIS epidemic model, the adoption of network neighbors encourages the consumer to adopt. In the new coupled model, the opinion of the consumer modifies the impact of adoption in (2) and encourages adoption via (3). Having briefly explored the product spread model in isolation, we introduce two different models of opinion dynamics.

III. Abelson Opinion Dynamics

The first opinion dynamic model that will be considered in conjunction with the product spread model in (1) is the canonical Abelson model, which in the 1960s laid the foundation for the study of opinion dynamics [9]. The modified dynamics follow

$$\begin{aligned}\dot{x}_i &= f_i(x,o)\\ \dot{o}_i &= g_i(x,o) = \sum_{\mathcal{N}_i^O} w_{ij}^o(o_j - o_i) + (x_i - o_i)\,,\end{aligned} \tag{5}$$

where $w_{ij}^o \geq 0$ is the weight on the opinion network. In the following discussion, it is assumed that $w_{ij}^o = 1,\ \forall i,j$. The last term of (5) moves an agent's opinion toward its adoption state. Hence, an agent's opinion is affected by its neighbor's opinions and its own adoption level.

As the proper behavior of the product spread model is dependent on the behavior of the opinion model, we show well-posedness of the opinion in the combined model.

Proposition 1: For the model in (5), if $x(0) \in [0,1]^N$ and $o(0) \in [0,1]^N$, then $x_i(t), o_i(t) \in [0,1]$ for all $t \geq 0$.

Proof: If $o_i(0) = 0$, $x(0) \in [0,1]^N$, and $o_j(0) \in [0,1]$ for all $j \neq i$, then by (5), $\dot{o}_i(0) \geq 0$, driving $o_i(t) \geq 0$ for $t > 0$. If $o_i(0) = 1$, $x(0) \in [0,1]^N$, and $o_j(0) \in [0,1]$ for all $j \neq i$, then by (5), $\dot{o}_i(0) \leq 0$, driving $o_i(t) \leq 1$ for $t > 0$.

Since there exists a derivative by (5), $o_i(t)$ is continuous. Therefore since we assume $o_i(0) \in [0,1]$ for all i, and have shown that for t such that $o_i(t) = 1$, $\dot{o}_i(t) \leq 0$ and for t such that $o_i(t) = 0$, $\dot{o}_i(t) \geq 0$, we have $o_i(t) \in [0,1]$ for all $t \geq 0$.

As the preceding argument holds for all $i \in \{1, 2, \ldots, N\}$ this implies $o(t) \in [0,1]^N$. By applying Lemma 1, we have that $x(t) \in [0,1]^N$. ■

We focus our attention on two equilibrium points which can be found by inspection, $z^* \in \{0_{2N}, 1_{2N}\}$, i.e. the equilibrium is either no one adopts the product and everyone has an opinion equal to zero, or everyone adopts the product and has an opinion equal to one. Discussion of the local stability of these equilibria requires analysis of the Jacobian of the opinion dependent product spread model, which turns out to be a Metzler matrix. A matrix, A, is said to be Metlzer if all off-diagonal elements are non-negative $a_{ij} \geq 0\ \forall i \neq j$.

If the Jacobian evaluated at an equilibrium is Hurwitz then the equilibrium is locally stable. We recall the following condition for Metzler matrices from [16]:

Lemma 2: For a Metzler matrix $A \in \mathbb{R}^{n\times n}$ the following statements are equivalent :

- A is Hurwitz
- There exists a $\xi \in \mathbb{R}^n$ such that ξ is element-wise positive and $A\xi$ is element-wise negative.

With Lemma 2 we can now discuss the local stability of two of the equilibria of the combined opinion-product spread model.

Theorem 1: The equilibrium point $z^* = 1_{2N}$ is locally stable if $\forall i,\quad \sum_{\mathcal{N}_i^P} \beta_{ij} + \beta_{ii} > \delta_i$ and equilibrium point $z^* = 0_{2N}$ is locally stable if $\forall i,\quad \delta_i > \sum_{\mathcal{N}_i^P} \beta_{ij} + \beta_{ii}$.

Proof: The Jacobian of the dynamics can be written in block form as:

$$J(z) = \left[\begin{array}{c|c} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial o} \\ \hline \frac{\partial g}{\partial x} & \frac{\partial g}{\partial o}\end{array}\right].$$

The first N rows of the Jacobian are are governed by (2) - (4). The remaining entries of the Jacobian follow:

$$\begin{aligned}\frac{\partial g_i}{\partial x_i} &= 1\\ \frac{\partial g_i}{\partial x_j} &= 0\ \forall j \neq i\\ \frac{\partial g_i}{\partial o_i} &= -d_i^O - 1\\ \frac{\partial g_i}{\partial o_j} &= \begin{cases} 1 & \text{if } j \in \mathcal{N}_i^O\ \forall j \neq i\\ 0 & \text{if } j \notin \mathcal{N}_i^O \cup \{i\},\end{cases}\end{aligned}$$

where d_i^O is the (in)degree of node i in the opinion network. Consider the Jacobian matrix at the equilibrium point $z^* = 1_{2N}$

$$J(z^*) = \left[\begin{array}{c|c} \text{diag}\left(-\sum_{\mathcal{N}_i^P} \beta_{ij} - \beta_{ii}\right) & \text{diag}\left(\delta_i\right) \\ \hline I & -(\mathcal{L}_O + I)\end{array}\right], \tag{6}$$

where $\mathcal{L}_O$ is the graph Laplacian of the opinion network. Note that the Jacobian in (6) is a Metzler matrix, allowing the application of Lemma 2.

To show the local stability of $z^* = 1_N$, consider $\xi = [1_N, \alpha 1_N]^T$ then $J(z^*)\xi$ results in $2N$ equations of the form

$$\begin{gathered}\left(-\sum_{\mathcal{N}_i^P} \beta_{ij} - \beta_{ii}\right) 1 + \delta_i\,(\alpha 1)\\ 1 - \alpha d_i^O - \alpha + \alpha \sum_{\mathcal{N}_i^O} 1\end{gathered} \tag{7}$$

By setting the equations in (7) less than 0, the following conditions appear

$$\alpha > 1$$

and

$$\sum_{\mathcal{N}_i^P} \beta_{ij} + \beta_{ii} > \alpha\delta_i > \delta_i$$

for all $i \in \{1, \ldots, N\}$. Therefore if $\sum_{\mathcal{N}_i^P} \beta_{ij} + \beta_{ii} > \delta_i\ \forall i$ then $\exists \alpha > 1$ such that $\xi = [1_N, \alpha 1_N]^T$ shows the Jacobian is Hurwitz, and $z = 1_{2N}$ is a locally stable equilibrium.

Now consider the Jacobian matrix at the equilibrium $z^* = 0_{2N}$

$$J(z^*) = \left[\begin{array}{c|c} \text{diag}\left(-\delta_i\right) & \text{diag}\left(\sum_{\mathcal{N}_i^P} \beta_{ij} + \beta_{ii}\right) \\ \hline I & -(\mathcal{L}_O + I)\end{array}\right],$$

Following a similar argument shows that the vector $\xi = [1_N, \alpha 1_N]^T,\ \alpha > 1$ leads to the condition $\delta_i > \sum_{\mathcal{N}_i^P} \beta_{ij} + \beta_{ii}$ for local stability of $z = 0_{2N}$. ■

The stability conditions for the equilibria is similar to the standard epidemic threshold for stability of the disease-free equilibrium, $\lambda_{\max}(BA - \text{diag}(\delta_i)) < 0$, where B is the matrix of β_{ij} (see Lemma 2 of [17] for a proof). Though the long term behavior of these systems has yet to be studied, the structure of the epidemic threshold suggests that convergence to these equilibria will depend on network structure. Simulations have shown that for a connected and undirected opinion graph, if $\forall i\ \sum_{\mathcal{N}_i^P} \beta_{ij} + \beta_{ii} > \delta_i$ then the system converges to $z = 1_{2N}$. However if $\exists i$ such that $\sum_{\mathcal{N}_i^P} \beta_{ij} + \beta_{ii} < \delta_i$ while $\sum_{\mathcal{N}_k^P} \beta_{kj} + \beta_{kk} > \delta_k\ \forall k \neq i$ then the equilibrium depends on the initial conditions.

The Abelson model has been extended in a number of ways which fit easily in the proposed coupling of opinion dynamics and product spread. Two examples are the bounded confidence model and the Altafini model. While these extensions are not the main focus of this paper, we will shortly comment on their effects on the product spread model.

First, the bounded confidence modification of the classical Abelson model, [18], [19], coupled with the product spread dynamics is

$$\dot{o}_i = g_i(x, o) = \sum_{\mathcal{N}_i^O} p(o_j, o_i)(o_j - o_i) + x_i - o_i. \quad (8)$$

where

$$p(o_j, o_i) = \begin{cases} w_{ij}^o & \text{if } \|o_j - o_i\| < \epsilon \\ 0 & \text{if else.} \end{cases}$$

The equilibria $z^* \in \{0_{2N}, 1_{2N}\}$ are also equilibria of this model and in fact in a small region around these equilibria the bounded confidence model is equal to Abelson dynamics giving the same characterization for local stability of equilibria. Simulations show that though the models share a set of equilibria, the behavior to reach those equilibria is different.

Second, the Altafini model [13] is of the form

$$\dot{o}_i = \sum_{\mathcal{N}_i^{\bar{O}}} |a_{ij}|(\text{sgn}(a_{ij}) o_j - o_i), \quad (9)$$

where $\mathcal{N}_i^{\bar{O}}$ is a signed set, with negative edges for the neighbors node i distrusts. It is well known that if the opinion graph is structurally balanced[1], then it can give a bipartite consensus, meaning all the members of one group converge to a value and all the members of the other group converge to the negative of that value [13], [20]. Alternatively, if the graph is structurally unbalanced then the opinions converge to 0_N [13], [21]. Due to this behavior the assumption of Lemma 1, that $o(t) \in [0, 1]$ for all $t \geq 0$, is difficult to meet. Therefore we need to slightly modify the model for $x(t)$ when employing Altafini-type dynamics. The complete model we propose is

$$\begin{aligned} \dot{x}_i &= f_i(x, \bar{o}) \\ \dot{o}_i &= \sum_{\mathcal{N}_i^{\bar{O}}} |a_{ij}|(\text{sgn}(a_{ij}) o_j - o_i) + x_i - \bar{o}_i, \end{aligned} \quad (10)$$

where $\bar{o}_i = o_i + .5$, we assume $o_i(0) \in [-.5, .5]\ \forall i$, and the notation in (10) is the same as in (9). Note that when there are no negative edges this reduces to the Abelson model in (5). When negative edges are present and the graph is structurally balanced the system can converge to a split equilibrium, that is, some nodes are completely infected and some nodes are completely healthy. This is illustrated via simulation in Section V. The behavior of the model is difficult to compare to the other models herein, because they do not have signed edges. We do include a simulation showing a comparison of the three models in this Section V. Given the negative edges in the A matrix, the equilibria set of the model becomes more complicated. Therefore we leave analysis of this model as an area for future investigation.

The Abelson opinion dynamics of this section induce the outcomes all adopt or all not-adopt. This reflects scenarios where a new technology or idea either becomes the new standard or completely fails to get adopted. [2]

IV. Threshold-Weighted Average Opinions

Threshold based models are fundamental models in the study of spreading processes over networks, having been considered since the late 1970s [22], [23]. However these models have yet to be studied in the setting of opinion exchange dynamics. We propose a model of opinion dynamics where individuals update their opinions using a weighted average of the opinions and product adoptions of friends, combined with a threshold. The threshold represents how stubborn or receptive one is to the influence of neighbors. As will be seen in the following analysis, this allows for polarization in opinions, resulting in coexistence of adopters and non-adopters.

Consider opinion dynamics defined by

$$\dot{o}_i = g_i(x, o) = o_i(1 - o_i)\left(h_i(x, o) - \tau_i\right), \quad (11)$$

where

$$h_i(x, o) = \frac{\sum_{\mathcal{N}_i^O} w_{ij}^o o_j + \sum_{\mathcal{N}_i^P} w_{ij}^x x_j}{\sum_{\mathcal{N}_i^O} w_{ij}^o + \sum_{\mathcal{N}_i^P} w_{ij}^x}.$$

The $w_{ij}^o \in [0, 1]$ represents node i's valuation of node j's opinion, and the $w_{ij}^x \in [0, 1]$ represents the influence j's adoption decision has over i's opinion. The opinion threshold, $\tau_i \in [0, 1]$, is a measure of stubbornness to opinion change. If $\tau_i = 1$, no amount of influence will force an increase in o_i. However, if $\tau_i = 0$, any amount of influence increases o_i.

We consider the well-posedness of the combination of the model in (1) and (11).

Proposition 2: In the opinion dynamics defined by (11), if $o(0) \in [0, 1]^N$ then $x(t), o(t) \in [0, 1]^N$ for all $t \geq 0$.

Proof: For any $s \geq 0$, if $o_i(s) = 0\ (= 1)$ then $o_i(t) = 0\ (= 1)$ for all $t \geq s$. Hence, by continuity of the o_i, for $o_i(0) \in [0, 1]$, $o_i(t) \in [0, 1]$ for all $t \geq 0$. As $o(t) \in [0, 1]^N$ then by Lemma 1 $x(t) \in [0, 1]^N$ for all $t \geq 0$. ■

The form of $\dot{o}_i$ in (11) is motivated from the replicator equation in evolutionary game theory. It follows a similar logistic growth and decay which depends on how a strategy's

[1] A signed graph is *structurally balanced* if it has a bipartition of the nodes V_1, V_2, i.e., $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$, such that $a_{ij} \leq 0$, $\forall v_i \in V_p$, $v_j \in V_q$ where $p, q \in \{1, 2\}$, $p \neq q$; otherwise, $a_{ij} \geq 0$ [13], [20].

[2] Examples of a new innovation being widely adopted are the invention of the steam engine and the administration of antibiotics. The practice of boiling water is an example of an innovation that failed to spread in the Peruvian village of Los Molinas, due to the inhabitants viewing it as incompatible with cultural beliefs. [1].

density-dependent fitness compares to the average fitness of the population.

A class of equilibria of the dynamics has the form

$$\mathcal{E} = \{z^* : x_i^* = o_i^* \in \{0, 1\}\},$$

i.e. in equilibrium, an agent adopts the product if and only if its opinion of the product is maximal, and avoids the product if and only if its opinion is zero.

Theorem 2: The locally stable equilibria $\mathcal{E}^* \subset \mathcal{E}$ of (1), (11) are given by

$$\mathcal{E}^* = \left\{ z^* \in \mathcal{E} : z_i^* = \begin{cases} 1 & \text{if } h_i(z^*) > \tau_i \\ 0 & \text{if } h_i(z^*) < \tau_i \end{cases} \right\}. \tag{12}$$

Essentially, the stable equilibria exhibit clustering in the network - a node adopts the product only if a sufficient number of its neighbors also adopt, and it does not adopt only if enough neighbors also do not adopt. Thus, there will be no "isolated" nodes in equilibrium, i.e. a node that adopts the product necessarily has neighbors who also adopt.

Proof: In order to examine the Jacobian of the opinion-dependent product spread model in (11), we compute the following quantities:

$$\frac{\partial g_i}{\partial x_i} = 0$$

$$\frac{\partial g_i}{\partial x_j} = \begin{cases} \dfrac{w_{ij}^x}{\sum_{\mathcal{N}_i^O} w_{ij}^o + \sum_{\mathcal{N}_i^P} w_{ij}^x} o_i(1 - o_i) & \text{if } j \in \mathcal{N}_i^P \ \forall j \neq i \\ 0 & \text{if } j \notin \mathcal{N}_i^P \cup \{i\} \end{cases}$$

$$\frac{\partial g_i}{\partial o_i} = (1 - 2o_i)(h_i(z) - \tau_i) \tag{13}$$

$$\frac{\partial g_i}{\partial o_j} = \begin{cases} \dfrac{w_{ij}^o}{\sum_{\mathcal{N}_i^O} w_{ij}^o + \sum_{\mathcal{N}_i^P} w_{ij}^x} o_i(1 - o_i) & \text{if } j \in \mathcal{N}_i^O \ \forall j \neq i \\ 0 & \text{if } j \notin \mathcal{N}_i^O \cup \{i\}. \end{cases}$$

For any equilibrium $z^* \in \mathcal{E}$, using the above and (2)-(4), the Jacobian takes the form

$$J(z^*) = \left[\begin{array}{c|c} \text{diag}\left(\frac{\partial f_i}{\partial x_i}(z^*)\right) & \text{diag}\left(\frac{\partial f_i}{\partial o_i}(z^*)\right) \\ \hline 0_{N \times N} & \text{diag}\left(\frac{\partial g_i}{\partial o_i}(z^*)\right) \end{array} \right].$$

Since this is a block upper-triangular matrix, the eigenvalues of $J(z^*)$ are simply

$$\text{eig}(J(z^*)) = \left\{ \frac{\partial f_i}{\partial x_i}(z^*) \right\}_{i=1}^{N} \cup \left\{ \frac{\partial g_i}{\partial o_i}(z^*) \right\}_{i=1}^{N},$$

and (2) gives that $\frac{\partial f_i}{\partial x_i}(z^*) < 0$. For local stability, we need all of the $\frac{\partial g_i}{\partial o_i}(z^*)$ defined by (13) to be negative. Therefore the conditions for stability are given by (12). ■

The novel opinion dynamics in this section allow all adopt or not-adopt as stable equilibria as in Section III, but also encourages new stable equilibria exhibiting segregation in the population between adopters of the product and non-adopters. This reflects scenarios where a new innovation diffuses to a subset of the population that are "like-minded", and fails to spread to the rest.

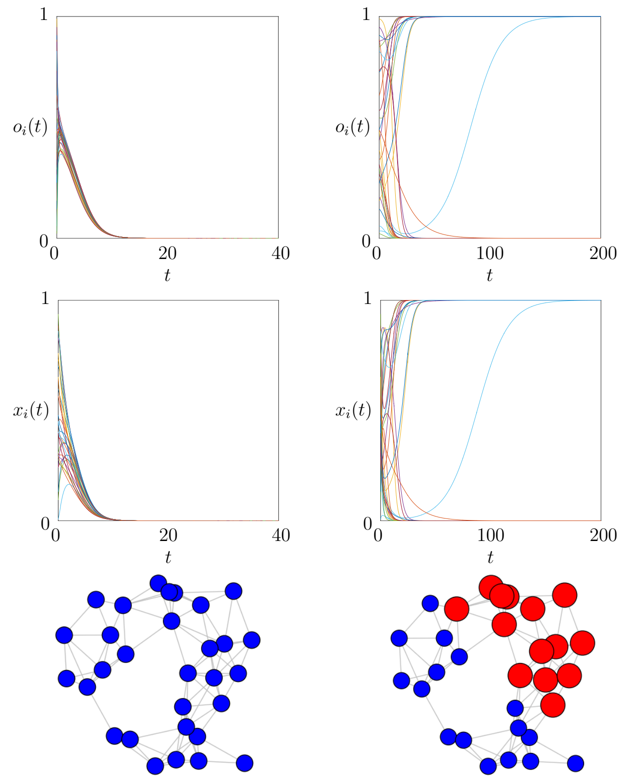

Fig. 1. Dynamics of the Abelson coupled model (left column) and threshold-based model (right column). In each model, all individual opinions $o_i(t)$ (top row) and adoptions $x_i(t)$ (middle row) are shown converging to their equilibrium values o_i^*, $x_i^* = 0$ or 1. The bottom row depicts the final equilibrium layout over the 30 node geometric network. Large nodes correspond to $x_i^* = 1$ and small nodes correspond to $x_i^* = 0$. The largest diameters indicate $o_i(0) = 1$ and the smallest diameters indicate $o_i(0) = 0$. The network for the SIS dynamics is depicted by the gray (positive) edges. For a video of this simulation please see youtu.be/U0bWaXCeayY.

V. Simulation Results

Having analyzed the local behavior of the proposed models, we examine the behavior of these models via simulation. Figure 1 shows a representative simulation of the Abelson and threshold-based dynamics. In this run, the Abelson dynamics quickly converge to the all not-adopt consensus. The threshold-based dynamic takes longer to converge to a stable equilibrium in $\mathcal{E}^*$, given by (12). We note that the final equilibrium outcome is heavily dependent on the initial opinions, as there are many possible stable fixed points the dynamics could converge to. We observe individuals whose opinion $o_i(t)$ changes directions before finally settling at either $o_i^* = 0$ or 1. The simulation of the two models is run on an undirected, unweighted geometric random network with thirty nodes, serving as both the opinion and product network, $\mathcal{G}_O$ and $\mathcal{G}_P$. The parameters are $\delta_i = 1, \beta_{ij} = .15, \tau_i = .5$ for all nodes i, j, and the max eigenvalue of the network is $\lambda_{\max} = 6.8146$. The initial condition of the simulations, chosen uniformly at random on $[0, 1]^{2N}$, are the same for both models.

The behavior of this coupled system leads the question: does influence from the adoption or opinion network drive the dynamics, or do they drive each other? Uncoupled from

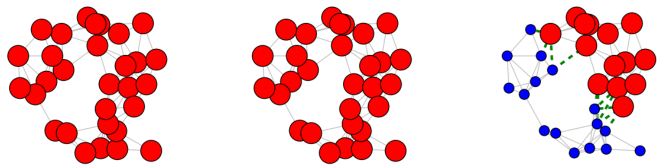

Fig. 2. The equilibria of simulations employing the Abelson model (Left), the Bounded Confidence model (Middle), and the Altafini model (Right) with the same conditions as the simulations in Figure 1 except dotted lines indicate negative edges, $\beta = .5$, and the confidence parameter $\epsilon = .1$: large nodes correspond to $x_i^* = 1$ and small nodes correspond to $x_i^* = 0$. The largest diameters indicate $o_i(0) = 1$ and the smallest diameters indicate $o_i(0) = 0$. The network for the SIS dynamics is depicted by the gray (positive) edges. For a video of this simulation please see youtu.be/BXVidqntYtA

the opinion dynamics, the adoption state $x(t)$ would converge to its endemic equilibrium $x^* \succ 0$, since the parameters satisfy the condition for endemic stability, $\delta/\beta < \lambda_{\max}$ [5]. Without opinions, each node reaches an intermediate value of adoption x_i^* whose value depends on its position in the network. When coupled with opinions for both Abelson and threshold-based dynamics, the $x_i(t)$ are driven to either $x_i^* = 0$ or 1, with their final opinions agreeing with their final adoption decisions. Given the difference in possible equilibria outcomes between the two models, the coupled opinion-adoption model is sensitive to the choice of opinion dynamic. Thus, opinions have a significant role in determining the final adoption state.

The final state of simulations of the bounded confidence (8) and Altafini models (9), variants of the Abelson opinion dynamics, are shown in Figure 2. The coupled bounded confidence dynamics converge to the all adopt equilibrium, exhibiting the same behavior as the Abelson model. This is surprising to observe, considering that bounded confidence opinion dynamics tend to form clusters of many different opinion levels. The coupled Altafini dynamics can exhibit final behavior similar to the threshold-based opinion model. However, static negative edges must be specified to attain such polarization. Hence while the threshold model has the possibility to reveal structure in a network, the Altafini model requires structure to be explicitly defined.

Together Figure 1 and Figure 2 show that the two primary opinion models considered in this paper are sufficient to influence the outcome of the product spread away from the endemic state and to capture diverse equilibrium outcomes, even over a simple graph.

VI. Conclusion

We have presented a framework that models opinion-dependent product spread over a network of agents. To ground the framework, we studied two distinct opinion spread dynamics and their effect on the population's propensity to adopt the product. The choice of opinion dynamic determines the stable outcomes of the coupled system, which are qualitatively different between the two models. In Section III, when opinions follow Laplacian consensus-based dynamics, the entire population either becomes adopters or non-adopters. In the threshold-based opinions of Section IV, cliques of adopters and non-adopters can form. In the case where product adoption run in isolation would have reached an endemic adoption state, we find that the coupled adoption state reaches a different equilibrium as the infection parameters have been modified by the opinion. This leads us to conclude that the opinion dynamics drive the equilibrium state of the system.

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