

Going Viral: Stability of Consensus-Driven Adoptive Spread

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Abstract—The spread of new products in a networked population is often modeled as an epidemic. However, in the case of “complex” contagion, these models do not capture nuanced, dynamic social reinforcement effects in adoption behavior. In this paper, we investigate a model of complex contagion which allows a coevolutionary interplay between adoption, modeled as an SIS epidemic spreading process, and social reinforcement effects, modeled as consensus opinion dynamics. Asymptotic stability analysis of the all-adopt as well as the none-adopt equilibria of the combined opinion-adoption model is provided through the use of Lyapunov arguments. In doing so, sufficient conditions are provided which determine the stability of the “flop” state, where no one adopts the product and everyone’s opinion of the product is least favorable, and the “hit” state, where everyone adopts and their opinions are most favorable. These conditions are shown to extend to the bounded confidence opinion dynamic under a stronger assumption on the model parameters. Additional analysis is provided for the case where the product is neither a hit nor a flop. To conclude, numerical simulations demonstrate behaviors indicated in the sociology literature such as tipping points.

1 INTRODUCTION

How technologies, behaviors, and ideas spread is a central question in the study of social human behavior. The spread of innovations (which we also refer to as products) has been studied since the work of Tarde [1] and is currently an area of active research, see for example [2]–[4]. Understanding the complex spreading process of innovations has important ramifications, including the economic impacts [4], [5] as well as the potential benefits that could come from spreading healthy behaviors [3], [6]–[8].

Innovation diffusion has often been studied through viral epidemic models [9], [10], in which those that have a product will spread it to others after a single contact with some probability [11]–[14]. This conceptual linkage between products and viruses has found its way into popular culture, resulting in companies hoping that their products will go “viral” and spread throughout the population. However, it has been shown that there are types of spread, specifically those that require social reinforcement or multiple

contacts, which are not well captured by these “simple” epidemic models [3], [15], [16]. For example, knowledge of a job opening spreads more widely through a network of acquaintances [17] while unproven or risky technology requires social validation before adoption [18], [19]. As many behavioral diffusion processes are known to depend on social interaction [5], [7], [8], this has led to the study of “complex” contagions [3], which take these peer effects into account.

This paper studies a networked dynamical system that highlights the complex contagion of a single product. The basis of our model is a coupling between a viral epidemic process and an opinion dynamic. This coupling captures the social reinforcement effects that create complex contagions. In particular, we use a modified SIS (susceptible-infected-susceptible) epidemic process [9], [20]–[29] to describe the evolution of product adoptions in the network. In an SIS epidemic process, agents can transition from a susceptible (non-adopting) to infected (adopting) state by contracting the virus from a neighbor. Infected agents can heal and become susceptible again. Also underlying our proposed system is the exchange of opinions in a social network, which may or may not coincide with the network in which product adoptions propagate. In particular, our SIS process co-evolves with a consensus protocol [30]–[36].

We interpret agents as being able to alternate between adopt and non-adopt states over time. Hence, the SIS process is an appropriate modeling consideration in contrast to SIR models [37], [38], where once an agent drops the product, they cannot re-adopt. Furthermore, consensus opinion dynamics are typically used to describe the exchange of opinions in a networked setting. Such processes were first investigated in mathematical sociology [31], [32] and has since been explored extensively for control applications [30], [33]–[36], [39].

1.1 Related works

A conceptually similar notion is the bi-virus epidemic model which studies two competing SIS viruses [40]–[46]. In these studies, the presence of a second type of virus influences the dynamic spread of the other, and vice versa. Furthermore, they spread on separate layered networks [44]–[46]. These features are common to our proposed model. However, the model presented in this paper is distinct in the coupling between different process dynamics associated with each

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layer, as opinion dynamics influence the spread of product adoptions, and vice versa.

Researchers in sociology have considered complex contagions through threshold models [12], [15], [47], [48] or conceptual extensions of epidemic models such as the independent cascade model [49], [50]. In a threshold model, an agent adopts if a given number [48] or fraction [51] of their neighbors has adopted the innovation. However, these dynamics fail to separate informational aspects of the adoption process (i.e. opinion exchange and dynamics) from actual acts of adoption. For instance, an individual may be satisfied in purchasing an Android phone even if all of his/her friends are satisfied with iPhones.

Indeed, there are few existing works that feature the coupling of the evolution of opinions and product adoption spread. In [13] a coupled adoption and awareness model is proposed that includes advertising. Similar to [20], this model assumes no graph structure, and models the system with only two differential equations, aggregating the population into two groups, susceptible (non-adopters) or infected (adopters). Introduced in [52], the Continuous Opinion Discrete Action (CODA) model considers discrete product adoption with Bayesian opinion updates, which do not depend on neighbors' opinions but only on the adoption actions of their network neighbors.

1.2 Our contributions

The specific contributions of this paper are listed below.

- The results from [53] are extended by analyzing the global behavior of the proposed model, providing conditions for the global stability of the none-adopt equilibrium and the all-adopt equilibrium.
- Our model is generalized to include a scaling factor that affects the sensitivity of changes in opinion to adoption behaviors.
- This paper includes an analysis of the system in the case of bounded confidence opinion dynamics [54], [55]. Similar results are derived in this case.
- We characterize a class of unstable equilibria. We show via simulation that these equilibria (when they exist in the interior of the state space) cause the evolution of the system to become initial-condition dependent.

Global stability analyses of our coupled system offers formal insights into a sub-field of sociology known as the "diffusion of innovations" [4], [5], [11], [12], [19], [56], which studies the spread of new products in society. These insights provide future opportunities to examine questions about control - what aspects of this dynamical process should be influenced, and how, in order to achieve desirable global behaviors.

1.3 Notation

For a vector $x(t)$, \dot{x} indicates its time derivative. We use 1_N and 0_N to indicate vectors in \mathbb{R}^N of all ones and zeros, respectively. The norm operator $\|\cdot\|$ is the Euclidean or 2-norm. For a matrix A , $\sigma(A)$ is the set of eigenvalues of A and $\alpha(A) := \max\{\text{Re}(\lambda) : \lambda \in \sigma(A)\}$. A diagonal matrix with its i th entry being x_i is denoted by $\text{diag}(x_i)$. We define $(0, 1)^N$ as the N Cartesian products of the interval $(0, 1)$ and $[0, 1]^N$ as the N Cartesian products of the interval $[0, 1]$.

2 MODEL

We modify the standard network dependent SIS epidemic ODE dynamics to incorporate the coupling between the "epidemic-like" spread of an innovation and opinion update dynamics. The adoption dynamics occur over a weighted, directed network \mathcal{G}_A of N nodes, or subpopulations. The opinion dynamics occur over a weighted digraph \mathcal{G}_O with the same node set as \mathcal{G}_A , but whose edges may or may not coincide with \mathcal{G}_A . The neighborhood set of node i is denoted as \mathcal{N}_i^X for $X = A, O$.

The possibility that the graphs \mathcal{G}_A and \mathcal{G}_O may differ reflects the group one discusses the product with can be distinct from the group one observes using the product. For example, an edge in the adoption graph which is not present in the opinion graph could reflect an agent observing a stranger using a new product.

Each node, or subpopulation, i has a proportion of agents that have adopted the product $x_i \in [0, 1]$. The subpopulation represented by node i also has an overall average opinion $o_i \in [0, 1]$, modeling how much the subpopulation values the product ($o_i = 0$ means the subpopulation is averse to the product, $o_i = 1$ means very receptive to the product).

Networked epidemic models have two different interpretations: the node state can correspond to 1) the probability of an individual being infected [9], or 2) the proportion of a subpopulation that is infected [21], [57]. In this paper we use the latter interpretation since adoption is a binary action, making this work distinct from the body of work that explores modeling adoption as a discrete process [12], [48], [52].

The adoption dynamics for each node evolve as a function of time:

$$\begin{aligned} \dot{x}_i &= f_i(x, o) \\ &\equiv -\delta_i x_i (1 - o_i) + (1 - x_i) o_i \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j + \beta_{ii} \right) \end{aligned} \quad (1)$$

where $\delta_i > 0$ is the drop rate for subpopulation i , $\beta_{ij} \geq 0$ is the exogenous adoption rate, and $\beta_{ii} \geq 0$ is the endogenous adoption rate. The parameters β_{ij} are the weights on the adoption graph while the parameter β_{ii} captures receptivity to opinion which can be influenced by other potential factors such as economic constraints. This model captures the connection between a subpopulation's opinion towards a product and their proclivity to adopt or to stop using a product. Here a node has an effective disadoption rate of $\delta_i(1 - o_i)$ and an effective adoption rate of $o_i \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j + \beta_{ii} \right)$.

Assumption 1. *It holds that $\beta_{ii} > 0, \forall i$.*

Assumption 1 ensures that should a subpopulation have a high opinion of the product, some of the subpopulation will adopt the product even if no network neighbors have adopted; i.e. that a subpopulation never ignores their opinion.

The primary opinion dynamic we consider in conjunction with the adoptive spread model in (1) is the canonical Abelson model, which in the 1960s laid the foundation for the study of opinion dynamics [31] and which has also been studied in the controls community as the consensus protocol

[58]. Here the opinion dynamic is treated as occurring at a subpopulation level, which is not commonly done in the opinion dynamics literature. The communities discussed in this case are conceptually similar to the marketing theory of brand communities [59], [60], where the community shares a common identity which impacts how they respond to a brand or product. Agents within the same community are expected to have similar opinions, as there is increased within group communication and a cohesive community self image. As such we here use the mean opinion to approximate the behavior of the group. Agents in the network can take part in multiple communities with regard to a given product [61], here we assume each agent is modeled as being part of their dominant community. The modified Abelson dynamics follow

$$\dot{o}_i = g_i(x, o) \equiv \sum_{j \in \mathcal{N}_i^O} w_{ij}^o (o_j - o_i) + w_i^x (\gamma_i x_i - o_i), \quad (2)$$

where $w_{ij}^o \geq 0$ is the weight on the opinion network, $w_i^x \geq 0$ is a weight that represents the responsiveness of the community to their opinion, and $\gamma_i \in [-1, 1]$ is a scaling factor describing response of a community to adoption. The scaling factor γ is influenced by product quality as well as how a product interacts with a community identity, i.e. a very good Android might not affect a community of iPhone users. The final term of the modified opinion dynamics, $w_i^x (\gamma_i x_i - o_i)$, describes the impact of a node's adoption behavior on its opinion.

The relationship between behavior and opinion, especially at the scale of a population, is difficult to capture and is likely highly nonlinear. Understanding the interplay between opinion and behavior is a central question in the social sciences, including such concepts as cognitive dissonance [62] and attitude-behavior consistency [63], [64] as well as many others. For the purpose of this initial exploration of the coupling of opinion and behavior, we draw inspiration from the model of Taylor [65], and treat the behavior of the population as a source of external information which influences the opinion of the population. Taken as a whole, (2) models the fact that a node's opinion is affected by its network neighbors' opinions and its own adoption level.

Assumption 2. *There exists i such that $w_i^x > 0$.*

Assumption 2 ensures that there is a coupling between the adoption state and the opinion somewhere in the network; essentially that there exists a node that cannot have adopted a product without having their opinion affected by the adoption. Some proofs require the stronger claim that $w_i^x > 0, \forall i$, which is reasonable given the interpretation of each node as a subpopulation defined by their reaction to a product.

Assumption 3. *It holds that $\gamma_i > 0, \forall i$.*

Assumption 3 restricts attention to the behavior of desirable products. One could alternatively consider a poorly designed product could have a negative impression on opinion, which is related to [66] where antagonism in the social network can cause negative opinions.

Assumption 4. *The opinion graph, \mathcal{G}_O is strongly connected.*

Writing the opinion dynamics into vector form,

$$\dot{o} = W\Gamma x - (\mathcal{L}_O + W)o, \quad (3)$$

where \mathcal{L}_O is the weighted in-degree graph Laplacian of the opinion network, $W = \text{diag}(w_i^x)$, and $\Gamma = \text{diag}(\gamma_i)$. The Laplacian $\mathcal{L}_O = D - A$ where $D = \text{diag}(d_i)$ is the in-degree matrix and A is the adjacency matrix of the opinion graph.

By combining (1) and (2), we have that the combined adoption-opinion dynamic follows

$$\begin{aligned} \dot{x}_i &= -\delta_i x_i (1 - o_i) + (1 - x_i) o_i \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j + \beta_{ii} \right), \\ \dot{o}_i &= \sum_{j \in \mathcal{N}_i^O} w_{ij}^o (o_j - o_i) + w_i^x (\gamma_i x_i - o_i). \end{aligned} \quad (4)$$

It is assumed the initial conditions $x_i(0), o_i(0) \in [0, 1] \forall i$ are known. As will be shown in the subsequent section, $x_i(0), o_i(0) \in [0, 1] \forall i$ implies $x_i(t), o_i(t) \in [0, 1] \forall i, t \geq 0$. Hence, $x_i(t)$ and $o_i(t)$ are functions from $[0, \infty)$ to $[0, 1]$. When convenient, we denote the aggregate state vector by $z = [x^T, o^T]^T$.

3 STABILITY ANALYSIS

In this section, we present the main results of the paper. These results include sufficient conditions for local and global stability of the none-adopt equilibrium 0_{2N} and the all-adopt equilibrium 1_{2N} . These two equilibrium points are foundational to understanding the behavior of this model. This section also considers analytically how the parameter γ affects the equilibria of the model as well as a case when there is an equilibrium $z^* \in (0, 1)^{2N}$.

For the coupled adoption opinion model in (4), each x_i represents the proportion of the i th subpopulation that has adopted the product and each o_i is a scaled average opinion of the i th subpopulation. Consequently, the proposed model is only meaningful for $x_i, o_i \in [0, 1]$. As such we first establish well-posedness of the model.

Proposition 1. *For the model in (4), if $z(0) \in [0, 1]^{2N}$, then $z(t) \in [0, 1]^{2N}$ for all $t \geq 0$.*

Proof. Observe that (4) is a system of polynomial ODEs over the compact state space $[0, 1]^{2N}$. This implies that the system of ODEs in (4) is Lipschitz on $[0, 1]^{2N}$ and as such the solutions $z_i(t)$ are continuous for all $i \in \{1, \dots, 2N\}$.

Suppose the proposition is not true. Then there is an index $i \in \{1, \dots, 2N\}$ such that $z_i(t)$ is the first state to go outside $[0, 1]$. Consider the case where $i \in \{1, \dots, N\}$, i.e. the adoption variable x_i leaves $[0, 1]$. If x_i becomes negative then there exists a time $s_0 > 0$ such that $x_i(s_0) = 0, \dot{x}_i(s_0) < 0, z_j(t) \in [0, 1] \forall t \in [0, s_0]$ and $\forall j \neq i$. However by (4),

$$\dot{x}_i(s_0) = o_i \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j + \beta_{ii} \right) \geq 0,$$

giving a contradiction. To show x_i cannot exceed one, we apply similar arguments and observe that

$$\dot{x}_i(s_0) = -\delta_i (1 - o_i) \leq 0.$$

This equality would contradict $\dot{x}_i(s_0) > 0$, which is required for x_i to exceed one.

Analogous arguments apply to show that when $i \in \{N+1, \dots, 2N\}$ z_i cannot leave $[0, 1]$, i.e. that o_{i-N} cannot go

below zero nor above one. In particular, if $\gamma_i \leq 1, \forall i$ then o_{i-N} can not go above one and if $\gamma_i \geq 0, \forall i$ then o_{i-N} can not go below zero. ■

Having shown the well-posedness of the adoption model, we now discuss properties of the adoptive spread model by considering the partial derivatives of the function in (1). Note

$$\frac{\partial f_i}{\partial x_i} = -\delta_i(1 - o_i) - o_i \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j + \beta_{ii} \right), \quad (5)$$

which is always negative when $z \in [0, 1]^{2N}$ since $\beta_{ij}, \delta_i \geq 0$ and $\beta_{ii} > 0$ by Assumption 1. The other set of partial derivatives with respect to x is

$$\frac{\partial f_i}{\partial x_j} = \begin{cases} (1 - x_i) o_i \beta_{ij} & \text{if } j \in \mathcal{N}_i^A, j \neq i \\ 0 & \text{if } j \notin \mathcal{N}_i^A \cup \{i\}, \end{cases}$$

which is always non-negative when $z \in [0, 1]^{2N}$ as $\beta_{ij} \geq 0$. We also have

$$\frac{\partial f_i}{\partial o_i} = \delta_i x_i + (1 - x_i) \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j + \beta_{ii} \right), \quad (6)$$

which is always non-negative when $z \in [0, 1]^{2N}$ since $\beta_{ij}, \delta_i \geq 0$. Finally,

$$\frac{\partial f_i}{\partial o_j} = 0 \quad \forall j \neq i. \quad (7)$$

As in the classic SIS epidemic model, the adoption of network neighbors encourages the consumer to adopt. In the new coupled model, the opinions of the consumers modify the impact of adoption in (5) and encourage adoption in (6).

Consider the behavior of the opinion dynamic model via the partial derivatives of the function in (2):

$$\frac{\partial g_i}{\partial x_i} = w_i^x \gamma_i \quad (8)$$

$$\frac{\partial g_i}{\partial x_j} = 0 \quad \forall j \neq i$$

$$\frac{\partial g_i}{\partial o_i} = -d_i^O - w_i^x$$

$$\frac{\partial g_i}{\partial o_j} = \begin{cases} 1 & \text{if } j \in \mathcal{N}_i^O \quad \forall j \neq i \\ 0 & \text{if } j \notin \mathcal{N}_i^O \cup \{i\}, \end{cases} \quad (9)$$

where d_i^O is the (in)degree of the i th node in the opinion network. Here the node's adoption state and the opinions of their network neighbors affect the opinion of the node.

This system always has an equilibrium at $z^* = 0_{2N}$, the case where no one adopts the product and everyone has an opinion equal to zero, which we refer to as the "flop" or "none-adopt" equilibrium. In the case of a high quality product, when $\gamma_i = 1, \forall i$, then $z^* = 1_{2N}$ is also an equilibrium point, i.e. everyone adopts the product and has an opinion equal to one, the "hit" or "all-adopt" equilibrium. If $\exists i, \gamma_i < 1$, then this affects the equilibria as shown in the following lemma.

Lemma 1. *If $\exists i$ s.t. $\gamma_i < 1$ and $w_i^x > 0$, then 1_{2N} is no longer an equilibrium point.*

Proof. Suppose node i satisfies $\gamma_i < 1$ and $w_i^x > 0$. Consider the opinion dynamic of node i at 1_{2N} .

$$\begin{aligned} \dot{o}_i &= \sum_{j \in \mathcal{N}_i^O} w_{ij}^o (o_j - o_i) + w_i^x (\gamma_i x_i - o_i) \\ &= w_i^x (\gamma_i x_i - o_i) < 0. \end{aligned}$$

Simulations have shown that if $\gamma_i < 1, w_i^x > 0 \quad \forall i$ there can exist a stable equilibrium point on $(0, 1)^{2N}$. At such equilibria, o_i^* is close to $\gamma_i x_i^*$, where close depends on the system parameters and underlying graph structures. The following Proposition describes one case where such an equilibrium can be analytically characterized. All stability proofs have been moved to the Appendix to facilitate exposition.

Proposition 2. *If the adoption graph is k -regular and $\forall i, j \gamma_i = \gamma < 1, \beta_{ij} = \beta, \beta_{ii} = \hat{\beta}, \delta_i = \delta$, such that $\gamma \hat{\beta} > \delta > 0$, then there exists an equilibrium point $z^* = [\tilde{x} 1_N; \tilde{o} 1_N]$ where $\gamma \tilde{x} = \tilde{o}$ and $\tilde{x} \in (0, 1)$.*

The parameter γ_i can also have an impact on the stability of 0_{2N} as shown by the following Proposition.

Proposition 3. *The equilibrium point $z = 0_{2N}$ is locally stable if $\forall i, \delta_i > \beta_{ii}$ or if $\forall i, w_i^x > 0, \delta_i > \gamma \beta_{ii}$, where $\gamma = \max_i \gamma_i$.*

We now move to characterizing the global stability of 0_{2N} and introduce the following notation

$$\Omega_i(\tau) \equiv \sum_{\mathcal{N}_i^A} \beta_{ij} \tau + \beta_{ii} \quad \text{for } i \in \{1, \dots, N\}, \quad (10)$$

which captures a node's maximal adoption rate when network neighbors have adoption $x_j \leq \tau$. We now state a main result characterizing the asymptotic stability of the flop equilibrium, 0_{2N} .

Theorem 1. *If $\delta_i > \Omega_i(\tau), w_i^x > 0, \forall i$, then 0_{2N} is asymptotically stable on $[0, \tau]^N \times [0, 1]^N$ for $\tau < 1$ and $[0, 1]^{2N} \setminus \{1_{2N}\}$ for $\tau = 1$.*

The proof relies on showing there is a quadratic Lyapunov function suitably defined on the subset in the statement. The parameter values that ensure global convergence, $\delta_i > \Omega_i(1) \quad \forall i$, can be interpreted as communities whose rate of disadoption is greater than any potential adoption from network neighbors. For example this can occur in a community whose identity does not allow for the use of a given product, such as a vegan community that will not consume a new meat product, even if all their network neighbors have adopted the product.

Remark 1. *The sufficient condition for stability of 0_{2N} on $[0, 1]^{2N} \setminus \{1_{2N}\}$ in Theorem 1 is equivalent to*

$$\Omega_i(1) - \delta_i < 0 \quad \forall i.$$

which by the Gershgorin Disc Theorem implies that

$$\alpha(B - D) < 0, \quad (11)$$

where B is the matrix of β_{ij} 's and $D = \text{diag}(\delta_i)$. This is the well-known necessary and sufficient condition for asymptotic stability of the healthy state 0_N for the general networked SIS epidemic model [44], [57]. Note that the condition in Theorem 1 causes all

the Gershgorin discs to be strictly in the left half plane, a sufficient condition for (11) to hold. Hence, the condition for Theorem 1 is more stringent than (11).

3.1 Behavior of Exceptional Products

This section considers the behavior of the system when an exceptional product, $\gamma_i = 1, \forall i$, is spreading through the network. First, the hit equilibrium $z^* = 1_{2N}$ is characterized and then a class of unstable equilibria is studied. We consider the behavior of the hit equilibrium $z^* = 1_{2N}$, which is an equilibrium point if $\gamma_i = 1, \forall i$.

Theorem 2. *If $\Omega_i(\tau) > \delta_i, w_i^x > 0, \gamma_i = 1 \forall i$, then 1_{2N} is asymptotically stable on $[\tau, 1]^N \times [0, 1]^N$ for $\tau > 0$ and $[0, 1]^{2N} \setminus \{0_{2N}\}$ for $\tau = 0$.*

When $\tau = 0$, the condition in Theorem 2 becomes $\beta_{ii} > \delta_i$ which captures that a sufficient level of innovation is required to ensure that a product takes off with no prior adoption.

Even in the case of an exceptional product, $\gamma_i = 1, \forall i$, if the stability conditions presented previously for the global stability of $z^* = 1_{2N}$ or $z^* = 0_{2N}$ are not satisfied, there is the possibility that an additional equilibrium point exists for the system. It is also possible that $z^* = 1_{2N}$ or $z^* = 0_{2N}$ are unstable. One class of these equilibria is identified in the following Lemma.

Lemma 2. *If $\gamma_i = 1 \forall i$ and there exists a z^* such that for all i it holds that $\delta_i = \sum_{\mathcal{N}_i^A} \beta_{ij} x_j^* + \beta_{ii}, x_i^* = o_i^*$ and $\sum_{j \in \mathcal{N}_i^o} (x_j^* - x_i^*) = 0$ then z^* is an equilibrium point. Furthermore, z^* is unique.*

Furthermore, this equilibrium, if it exists, is shown to be unstable.

Theorem 3. *If the equilibrium described in Lemma 2 exists and $w_i^x > 0, \forall i$, it is unstable.*

The results of Theorem 3 provide valuable insight into the behavior of this model. It is possible that the equilibrium in Lemma 2 exists in $(0, 1)^{2N}$. If such an equilibrium exists and the opinion graph is strongly connected then this equilibrium is unstable, causing the system behavior to be dependent on the initial condition. This behavior is explored in more detail in the Simulation Section. Furthermore, the equilibria at 0_N and 1_N can satisfy the properties of Lemma 2, providing instability conditions for the fundamental equilibrium points of the system. We summarize the stability conditions for these equilibria in the following Corollaries and in Table 1.

Corollary 1. *Suppose the opinion graph is strongly connected and $\gamma_i = 1, w_i^x > 0, \forall i$. If $\delta_i = \beta_{ii}, \forall i$, then 0_{2N} is unstable; if $\delta_i > \beta_{ii}, \forall i$, then 0_{2N} is locally stable; and if $\delta_i > \Omega_i(1), \forall i$, then 0_{2N} is asymptotically stable.*

Corollary 2. *Suppose the opinion graph is strongly connected and $\gamma_i = 1, w_i^x > 0, \forall i$. If $\delta_i = \Omega_i(1), \forall i$, then 1_{2N} is unstable; if $\Omega_i(1) > \delta_i, \forall i$, then 1_{2N} is locally stable; and if $\beta_{ii} > \delta_i, \forall i$, then 1_{2N} is asymptotically stable.*

Corollaries 1 and 2 also elucidate the expected behavior when the equilibrium in Lemma 2 exists in $(0, 1)^{2N}$. In such

z^*	Unstable	Local Stable	Asymptotic Stable
0_{2N}	$\delta_i = \beta_{ii}$	$\delta_i > \beta_{ii}$	$\delta_i > \Omega_i(1)$
1_{2N}	$\Omega_i(1) = \delta_i$	$\Omega_i(1) > \delta_i$	$\beta_{ii} > \delta_i$

TABLE 1: Summary of stability conditions: recall from (10) $\Omega_i(\tau) \equiv \sum_{\mathcal{N}_i^A} \beta_{ij} \tau + \beta_{ii}$.

a case, $\Omega_i(1) > \delta_i > \beta_{ii}$, which implies that 1_{2N} and 0_{2N} are both locally stable. We explore this further in the Simulation Section.

4 BOUNDED CONFIDENCE MODEL

In this section, we extend our consideration to the bounded confidence opinion dynamic model. The bounded confidence model is an extension of the Abelson opinion dynamic model [54], [55], which when coupled with the adoption dynamic is as follows:

$$\dot{o}_i = g_i(x, o) = \sum_{j \in \mathcal{N}_i^o} p(o_j, o_i)(o_j - o_i) + w_i^x(\gamma_i x_i - o_i), \quad (12)$$

where

$$p(o_j, o_i) = \begin{cases} w_{ij}^o & \text{if } \|o_j - o_i\| < \xi \\ 0 & \text{otherwise.} \end{cases}$$

Under the bounded confidence model, nodes will sever a link in the opinion graph if the nodes have sufficiently different opinions and maintain or reintroduce the link if the respective opinions are closer than ξ . This behavior is essentially a state dependent switch between opinion graph topologies. These opinion graphs may not be connected, to the point where each node may have no neighbors in the opinion graph. However the structure of the coupling with the adoption dynamic ensures that the conditions for the asymptotic equilibria $z^* \in \{0_{2N}, 1_{2N}\}$ are the same.

As is detailed in the Appendix, we consider arbitrary switching behavior in the bounded confidence model and as such the domain over which stability is proven must be modified from $[0, 1]^{2N} \setminus \{1_{2N}\}$ to $[0, 1]^{2N}$. It is likely that the specific interaction pattern of the bounded confidence model will allow stronger results to be shown under additional constraints, however such a consideration is left for future work. We have the following asymptotic stability results for the bounded confidence version of our model.

Theorem 4. *If $\delta_i > \Omega_i(1) \forall i$, the opinion graph is undirected, and $\Omega_i(1) = w_i^x, \forall i$ then 0_{2N} is uniformly asymptotically stable on $[0, 1]^{2N}$ under the bounded confidence opinion dynamic.*

Theorem 5. *If $\beta_{ii} > \delta_i, \gamma_i = 1 \forall i$, the opinion graph is undirected, and if $\delta_i = w_i^x, \forall i$, then 1_{2N} is asymptotically stable on $(0, 1]^{2N}$ under the bounded confidence opinion dynamic.*

Simulation have shown that the system exhibits asymptotic stability without the conditions which enforce symmetry for the matrices P_i , i.e. if the opinion graph is directed and $\Omega_i(1) \neq w_i^x, \forall i$, the system still seems to be asymptotically stable. The theoretical characterization of such a case will be left for future work.

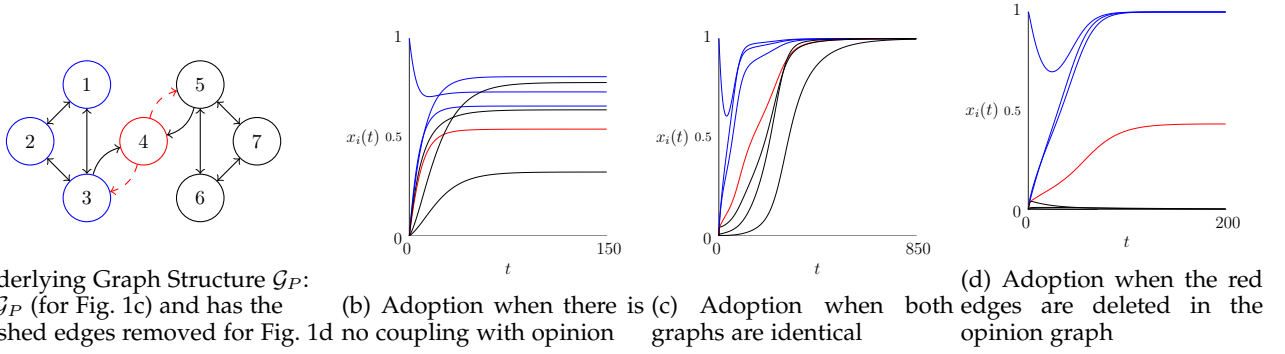


Fig. 1: The adoption of a barbell graph with random parameters run starting from $x_1 = 1$ and $x_j = 0 \forall j \neq 1$ for three separate conditions on the opinion graph. Figure 1b shows when there is no coupling with opinion (i.e. it is the standard SIS epidemic model). Figure 1c shows when the opinion graph is an unweighted version of the infection graph. Figure 1d shows when the opinion graph has the two red, dashed edges deleted from the adoption graph, making node 4 an information bottleneck.

5 SIMULATION

In this section, we consider the consensus-adoption model in (4) and show instances through numerical simulations where it exhibits behaviors that have been observed in empirical studies of adoptive spread [15], [67]. In particular, we focus on behaviors shown in the study of complex contagions, which aims to understand the effect of social reinforcement on adoption dynamics. We show that the model exhibits different complex contagion behavior under certain opinion topologies by highlighting the importance of weak ties. We also show parameter regimes where our model behavior is dependent on initial conditions, showing threshold-like behavior similar to that of a tipping point. We conclude the section by observing the existence of stable equilibria not described in the theoretical results.

5.1 Complex Contagion: The Impact of Opinion

In the sociology literature, and specifically the study of complex contagions, a distinction is made between the impact of strong and weak friendship ties in a social network [15], [17]. A weak tie is characterized by the lack of many common friendships and low emotional intensity, representing for example a tie with an acquaintance. A strong tie is the opposite, having high intimacy and strong emotional intensity, for example a tie with a family member or close friend. Granovetter found that for information diffusion, such as the availability of jobs, weak ties are vital for the spread of information by serving as bridges between different communities [17].

While weak ties are important for the spread of simple contagions, like information or a virus; in the case of a complex contagion these weak ties can serve as bottlenecks. This is due to the fact that complex contagions require social reinforcement to spread, which is often missing in the case of a weak tie [15]. The coupled adoption opinion model presented in (4) allows the study of the effect of the underlying graph structure of the opinion dynamic, which captures social reinforcement. Figure 1 shows that varying the underlying graph of the opinion dynamic can change a weak tie from a conduit to a bottleneck.

The coupled adoption opinion model is simulated on an example 7-node network with the adoption graph \mathcal{G}_P shown in Figure 1a. The opinion graph \mathcal{G}_O is varied between Figures 1b-1d, however across all opinion graphs the edge weights follow $w_{ij}^o = 1, \forall i, j \in \{1, \dots, n\}$. The randomly chosen adoption parameters are as follows:

$$B = \begin{bmatrix} 0.0665 & 0.0668 & 0.0630 & 0 & 0 & 0 & 0 \\ 0.0718 & 0.0033 & 0.0477 & 0 & 0 & 0 & 0 \\ 0.0281 & 0.0521 & 0.0549 & 0.0641 & 0 & 0 & 0 \\ 0 & 0 & 0.0114 & 0.0525 & 0.0480 & 0 & 0 \\ 0 & 0 & 0 & 0.0250 & 0.0646 & 0.0432 & 0.0575 \\ 0 & 0 & 0 & 0 & 0.0112 & 0.0050 & 0.0346 \\ 0 & 0 & 0 & 0 & 0.0470 & 0.0421 & 0.0108 \end{bmatrix},$$

$D = \text{diag}(0.0599, 0.0208, 0.0790, 0.0767, 0.0773, 0.0813, 0.0156)$, and $\gamma_i = 1, \forall i$. It holds that $\beta_{ij} < \delta_i, \forall i \neq 1$, suggesting that the population is unlikely to adopt the innovation without the influence of node 1. This system was initialized at $x(0) = [1, 0, 0, 0, 0, 0, 0]^T$ and $o(0) = [0.8279, 0.2410, 0.7215, 0.9841, 0.6457, 0.5573, 0.9630]^T$.

As can be seen from Figure 1, varying information topologies causes very different behavior in the model. If there is no coupling with an opinion, shown in Figure 1b, then the innovation spreads from node 1 to the entire population with varying success. If the adoption graph and the opinion graph are identical, as in Figure 1c, then the innovation is able to completely spread throughout the graph.

Figure 1d shows the impact of an information bottleneck: here node 4 receives information about the opinion of node 3 and node 5 but does not spread information as the outgoing links have been deleted. This stops information about the innovation from nodes 1 to 3 from spreading to nodes 5 to 7 but does not affect the adoption network. Consequently, the information bottleneck, node 4, is also an adoption bottleneck; i.e. the information topology prevents the product from spreading into the right half of the graph.

We have shown the impact that opinion can have on adoption behavior in the model presented in (4), suggesting that this model is a valuable tool for understanding the social reinforcement effects typically studied via threshold models.

5.2 Tipping Point-Like Behavior

A central concept in the spread of innovations is the tipping point [47], [48], [67]. A tipping point on a population level is a fraction of adopters which determines the prevalence of the product. If the adoption level is under the tipping point, the product does not spread to the whole population while if the adoption is above the tipping point the product spreads to the whole population. Lemma 2 describes an equilibrium point which occurs if the coupled adoption opinion model satisfies for all i that $\delta_i = \sum_{j \in \mathcal{N}_i^A} \beta_{ij} x_j^* + \beta_{ii}$, $x_i^* = o_i^*$ and $\sum_{j \in \mathcal{N}_i^O} x_j - x_i = 0$. Theorem 3 shows that this equilibrium is unstable if the opinion graph is strongly connected and $\gamma_i = 1, \forall i$. Simulations show that if such an unstable equilibrium exists in $(0, 1)^{2N}$, it will induce initial condition dependent behavior in the model, similar to a tipping point.

This dependence is shown in Figure 2. As a simple example which shows this behavior, the adoption graph \mathcal{G}_P is a complete graph on four nodes and the opinion topology \mathcal{G}_O is taken to be a complete graph with edge weights $w_{ij}^o = 1, \forall i, j$. The system parameters are

$$B = \begin{bmatrix} 0.1 & 0.25 & 0.3 & 0.35 \\ 0.15 & 0.05 & 0.3 & 0.3 \\ 0.5 & 0.3 & 0.1 & 0.3 \\ 0.2 & 0.1 & 0.1 & 0.2 \end{bmatrix},$$

$D = \text{diag}(.5, .4, .6, .3)$, and $\gamma_i = 1, \forall i$. With these parameters, $x_i = o_i = 0.5, \forall i$ is an equilibrium as described in Lemma 2. The system was run 10,000 times with randomly selected initial conditions under both the consensus and the bounded confidence model with $\xi = 0.01$. The results are shown in Figure 2. When the initial condition is sufficiently high, the system converges to the hit equilibrium, 1_{2N} . Conversely if the system has an initial condition that was sufficiently low, the system converges to flop equilibrium, 0_{2N} . In the bounded confidence case, one or more nodes can break away from their neighbors, resulting in a split equilibrium. Under the current system parameters, Figure 2 suggests that initial adoption has a larger impact on the resultant equilibrium. This matches what one might expect based on the conditions of Theorem 1 and 2, however further work is required to completely characterize the tipping point-like behavior of the system and how a system which shows these behaviors would react to control effort.

5.3 Stable Equilibrium

While this paper has primarily focused on the analytical characterization of the hit, the equilibrium at 1_{2N} , and the flop, the equilibrium at 0_{2N} , it is also possible that there exists a stable equilibrium in the open interval $(0, 1)^{2N}$. While the full characterization of such equilibria will require future work, it is possible to discuss their existence through simulation.

One way that these stable equilibria can exist is if there is a mix between nodes which satisfy $\delta_i > \Omega_i(1)$ and nodes which satisfy $\delta_i < \beta_{ii}$. Figure 3 considers a 5-node

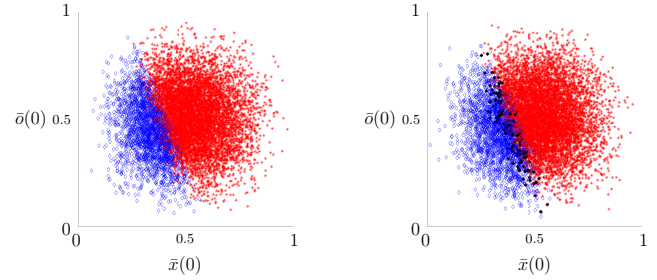


Fig. 2: Simulation of the effect of initial conditions on the system equilibria in the presence of an unstable equilibrium. The x-axis shows mean initial adoption, the y-axis mean initial opinion. The blue diamonds and red stars represent when the system converged to 0_{2N} and 1_{2N} , respectively. The right figure shows what happens when the bounded confidence model is run as the opinion dynamic, resulting in systems that do not converge to 0_{2N} or 1_{2N} , shown by black crosses.

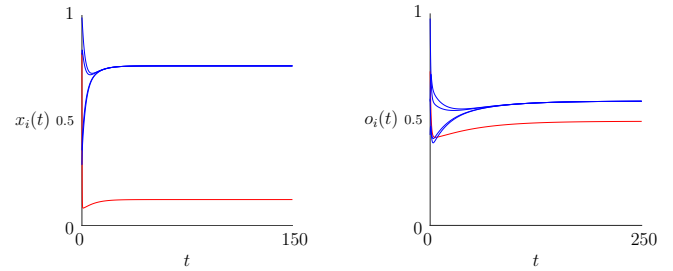


Fig. 3: System evolution, with adoption on the left and opinion on the right, on the star graph where there is a stable equilibrium in $(0, 1)^{2N}$. The red line represents the center node, while the blue lines represent the peripheral nodes.

star graph topology with node 1 as the center node. When $D = \text{diag}(5, 0.1, 0.1, 0.1, 0.1)$ and

$$B = \begin{bmatrix} 0.1 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.01 & 0.15 & 0 & 0 & 0 \\ 0.01 & 0 & 0.15 & 0 & 0 \\ 0.01 & 0 & 0 & 0.15 & 0 \\ 0.01 & 0 & 0 & 0 & 0.15 \end{bmatrix}$$

the system follows the trajectory in Figure 3. The center node satisfies $\delta_1 > \Omega_1$ while the peripheral nodes satisfy $\delta_k < \beta_{kk}, \forall k \neq 1$. Here δ_1 is large enough that the nodes will not converge to 1_{2N} but instead reach an equilibrium value in the interval $(0, 1)^{2N}$. Figure 3 shows the evolution of the system under a sample initial condition. The equilibrium at

$$x = \begin{bmatrix} 0.1114 & 0.682970.6829 & 0.6829 & 0.6829 \end{bmatrix}^T,$$

$$o = \begin{bmatrix} 0.4924 & 0.5877 & 0.5877 & 0.5877 & 0.5877 \end{bmatrix}^T$$

was shown to be unique by running the system 1000 times with varying initial conditions. Simulations show that under these parameters the hit and flop equilibria are both unstable.

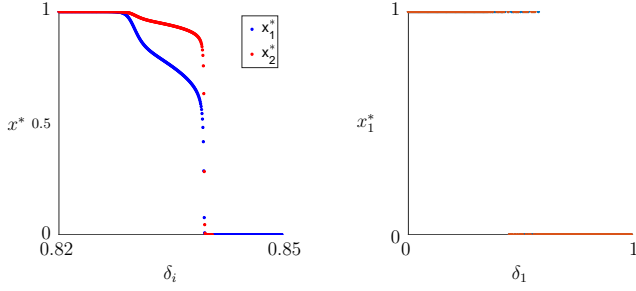


Fig. 4: Bifurcation Analysis of a two node system. On the left, the equilibrium adoption x^* when $\delta_2 = .25$ and the system is initialized at $x = 1_2 - \{10^{-6}\}_2, o = 1_2$. On the right, x_1^* when $\delta_2 = 0.45$ starting at various random initial conditions.

In order to understand in more depth how a stable equilibrium can exist which is not 1_{2N} or 0_{2N} we consider a completely connect two node system where $\delta_2 = 0.25$, and $B = \begin{bmatrix} 0.3 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}$. The δ_1 parameter is varied to see what happened when the system was initialized at $z = [1 - 10^{-6}; 1 - 10^{-6}; 1; 1]$. As can be seen in the left side of Figure 4 as the strength of δ_1 increases, $[1; 1]$ is no longer an equilibrium and an intermediate stable equilibrium exists. When the dis-adoption becomes too strong, the system is forced to 0.

In the case where the tipping point-like equilibrium exists (here when $\delta_2 = 0.45$), there seems to be no other equilibrium in $(0, 1)^{2N}$. The right side of Figure 4 shows the system simulated from a variety of random initial conditions, showing that the system always ends up at either 1_2 or 0_2 .

Another way this can happen is if $\gamma_i < 1, \forall i$ and the 0_{2N} equilibrium is unstable. Figure 5 shows the results for a 20-node Erdos-Renyi graph with link probability 0.3 where $\beta_{ii} = 0.15, \forall i$ and $\beta_{ij} = 0.005$ when a link exists between i and j , $D = \text{diag}(.1_N)$, the opinion graph is identical to the adoption graph but with unit weights, and $\gamma_i = 0.75, \forall i$ which was initialized at a random initial condition. At equilibrium $o_i^* \approx \gamma_i x_i^*, \forall i$. This extends the results of Proposition 2, showing that variations in graph structure impact the equilibrium point even if the model parameters are uniform.

6 CONCLUSION

In this paper, we consider a coupled adoption opinion model in which the spread of an epidemic product is influenced by the evolution of an opinion dynamic. The stability of the hit and flop equilibria are shown based on the adoption parameters and the opinion graph. These results are also extended to the case of the bounded confidence opinion dynamic. Finally, it was shown in simulation that the presented model exhibits many of the characteristic behaviors of product spread observed in the sociology literature.

This paper provides a dynamical systems viewpoint for the coupling between adoption and opinion, and ultimately provides an avenue to deepen the understanding of

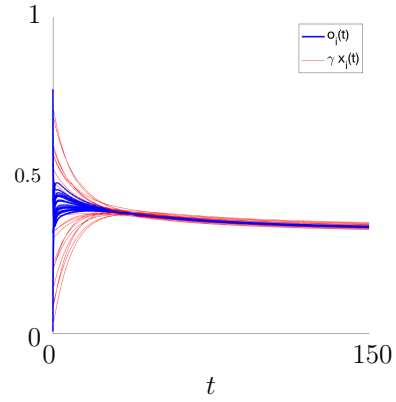


Fig. 5: System evolution, with opinion in blue and adoption multiplied by the parameter γ_i in red on an Erdos-Renyi graph where there is a stable equilibrium in $(0, 1)^{2N}$ caused by $\gamma_i < 1, \forall i$.

complex contagion adoption dynamics. Behavior observed from the sociology literature, such as tipping points, were exhibited by the model, pointing to its ability to capture a wide range of real world behaviors. Future work is required to explore and connect these behaviors to real contagion phenomenon and the possibility of applying control.

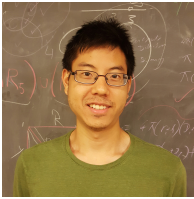
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APPENDIX

In this appendix, the proofs of the various technical results in the paper are presented. First the proof of Proposition 2 is presented before transitioning to the stability of the 1_{2N} and 0_{2N} equilibria for the original model presented in (4). The proofs that follow require some concepts from matrix analysis which are discussed below. Unless otherwise stated the discussion follows [68].

Proof of Proposition 2. We begin with the opinion dynamic in (2). Substituting in the point $z^* = [\tilde{x}1_N; \tilde{o}1_N]$ gives

$$\dot{o}_i = \sum_{j \in \mathcal{N}_i^O} w_{ij}^O (\tilde{o} - \tilde{o}) + w_i^x (\gamma \tilde{x} - \tilde{o}), \quad \forall i.$$

For this equation to equal zero we must have $\gamma \tilde{x} = \tilde{o}$.

To show that z^* is an equilibrium of the adoption dynamic, consider the following which is obtained by substituting the assumptions and $o_i = \gamma x_i$ into the adoption dynamics:

$$\dot{x}_i = -k\beta\gamma x^3 + (\delta + k\beta - \hat{\beta})\gamma x^2 + (-\delta + \hat{\beta}\gamma)x. \quad (13)$$

To see that $\dot{x}_i = 0$ for some $\tilde{x} \in (0, 1)$, consider \hat{x} at $x = \epsilon > 0$ and $x = 1$. At $x = \epsilon$, we have $\dot{x} \approx (-\delta + \hat{\beta}\gamma)\epsilon$, which is positive by assumption. At $x = 1$, the expression simplifies to $\dot{x} = \delta(\gamma - 1) < 0$, also by assumption. Therefore, since the right side of (13) is continuous and it is positive at $x = \epsilon$ and negative at $x = 1$, there must exist at least one $\tilde{x} \in (0, 1)$ such that $\dot{x}_i = 0$. As the parameters are uniform across agents, the \tilde{x} at which $\dot{x}_i = 0$ is the same for all i . ■

Definition 1. A matrix A is diagonally dominant if

$$|A_{ii}| \geq \sum_{j \neq i} |A_{ij}| \quad \forall i.$$

The matrix is strictly diagonally dominant if the inequality is strict.

Consider a diagonally dominant matrix A and let

$$J = \left\{ i \in \{1, 2, \dots, n\} : |a_{ii}| > \sum_{j \neq i} |a_{ij}| \right\}.$$

Any row j such that $j \in J$ is said to be a strictly diagonally dominant row.

A diagonally dominant matrix with negative diagonal entries has eigenvalues with non-positive real part by the Gershgorin Disc Theorem a strictly diagonally dominant matrix with negative diagonal entries has eigenvalues with negative real part by the Gershgorin Disc Theorem.

Definition 2. A matrix A is weakly chained diagonally dominant (WCDD) if it is

- diagonally dominant, and
- for all $i \notin J$ there is a sequence of nonzero elements of A of the form $a_{ii_1}, a_{i_1 i_2}, \dots, a_{i_r j}$ with $j \in J$.

The second condition can be equivalently expressed as the existence of a walk from i to j on the directed graph of A . WCDD matrices have the following characterization [69]:

Lemma 3. A WCDD matrix is nonsingular.

Recall the following condition for Metzler matrices from [70]:

Lemma 4. Let A be an irreducible Metzler Matrix. In the following, for two vectors $x, y \in \mathbb{R}^N$, $x > y$ means $x_i > y_i \forall i$.

- If there exists $x > 0_N$ such that $Ax > \lambda x$ for some $\lambda \in \mathbb{R}$, then $\alpha(A) > \lambda$.
- If there exists $x > 0_N$ such that $\mu x > Ax$ for some $\mu \in \mathbb{R}$, then $\mu > \alpha(A)$.

Lemma 5. (Proposition 1 and Lemma A.1 in [22]) Suppose that M is an irreducible Metzler matrix such that $\alpha(M) < 0$ ($\alpha(M) = 0$). Then, there exists a positive diagonal matrix Q such that $M^T Q + QM$ is negative definite (semi-definite).

With the mathematical preliminaries concluded, stability of the equilibrium point at $z^* = 0_{2N}$ can be shown.

Proof of Proposition 3. The Jacobian of the dynamics can be written in block form as:

$$J(z) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial o} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial o} \end{bmatrix},$$

where first N rows of the Jacobian are given by (5)-(7) and the second N rows are given by (8)-(9).

Consider the Jacobian matrix at the equilibrium point $z^* = 0_{2N}$:

$$J(z^*) = \left[\begin{array}{c|c} \text{diag}(-\delta_i) & \text{diag}(\beta_{ii}) \\ \hline W\Gamma & -(\mathcal{L}_O + W) \end{array} \right],$$

using the notation from (3). To show local stability, this Jacobian must be shown to be Hurwitz.

In the case that $\delta_i > \beta_{ii} \forall i$, together with the fact that $\gamma_i \leq 1, \forall i$ and the graph Laplacian is diagonally dominant, $J(z^*)$ is diagonally dominant. The first N rows of $J(z^*)$ are strictly diagonally dominant while the diagonal dominance of the second N rows of $J(z^*)$ depends on γ_i and w_i^x . If $\gamma_i = 1$ the row is diagonally dominant, if $\gamma_i < 1$ the row is strictly diagonally dominant. If $w_i^x = 0$ the row is diagonally dominant; otherwise, it is strictly diagonally dominant. However, by Assumption 2 $\exists j$ s.t. $w_j^x > 0$ and by Assumption 4 the opinion graph is strongly connected. Therefore there is a path from row j to all rows $k \in \{N+1, N+2, \dots, 2N\}$. So the Jacobian is WCDD and therefore nonsingular by Lemma 3.

Since the diagonal elements of $J(z^*)$ are negative, the above argument combined with the Gershgorin Disc Theorem shows that the Jacobian is Hurwitz.

If $\delta_i > \gamma\beta_{ii} \forall i$, there exists a $\rho > 1$ such that $\delta_i > (\rho\gamma)\beta_{ii} \forall i$. As the opinion graph is strongly connected the Jacobian is irreducible allowing the use of Lemma 4. Consider a vector y of the form $y = \begin{bmatrix} 1_N^T & \rho\gamma 1_N^T \end{bmatrix}^T$. Then consider the matrix product $J(z^*)y$. The first N entries of $J(z^*)y$ follow:

$$-\delta_i + \beta_{ii}\rho\gamma < 0.$$

The last N entries of $J(z^*)y$ follow

$$\gamma_i w_i^x - d_i^O \rho\gamma - w_i^x \rho\gamma + \sum_{N_i^O} \rho\gamma = (\gamma_i - \rho\gamma)w_i^x < 0.$$

Therefore by Lemma 4 the Jacobian is Hurwitz. ■

Having shown local stability of the flop equilibrium $z^* = 0_{2N}$, we move to showing asymptotic stability.

Before proving Theorem 1, we introduce some lemmas that will be required for the proof. Consider the matrix

$$P = \begin{bmatrix} -\bar{B} & \bar{B} \\ W\Gamma & -(\mathcal{L}_o + W) \end{bmatrix} \quad (14)$$

where $\bar{B} = \text{diag}(\Omega_i(\tau))$.

Lemma 6. *If $\delta_i > \Omega_i(\tau)$, $\forall i$ and the system is in $[0, \tau]^N \times [0, 1]^N$, then the coupled dynamic in (4) satisfies $\dot{z} \leq Pz$.*

Proof. Consider the adoption dynamic for the case where the adoption parameters satisfy $\delta_i > \Omega_i(\tau), \forall i$ and when the state satisfies $x_i \neq 0$ and $o_i \neq 1$:

$$\begin{aligned} \dot{x}_i &= -\delta_i x_i (1 - o_i) + (1 - x_i) o_i \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j + \beta_{ii} \right) \\ &\leq -\delta_i x_i (1 - o_i) + (1 - x_i) o_i \Omega_i(\tau) \\ &< -\Omega_i(\tau) x_i (1 - o_i) + (1 - x_i) o_i \Omega_i(\tau) \\ &= (o_i - x_i) \Omega_i(\tau). \end{aligned}$$

Consider now when $x_i = 0$ or $o_i = 1$. In the case that $o_i = 1$ then $\dot{x}_i \leq (1 - x_i) \Omega_i(\tau)$ and $(o_i - x_i) \Omega_i(\tau) = (1 - x_i) \Omega_i(\tau)$. In the case that $x_i = 0$ then $\dot{x}_i \leq o_i \Omega_i(\tau)$ and $(o_i - x_i) \Omega_i(\tau) = o_i \Omega_i(\tau)$. Together it holds that when $x_i = 0$ or $o_i = 1$

$$\dot{x}_i \leq (o_i - x_i) \Omega_i(\tau).$$

Translating this to matrix form gives that $\dot{z} \leq Pz$. ■

Lemma 7. *The eigenvalues of P all have non-positive real part. If $\exists i$, s.t. $\gamma_i < 1, w_i^x > 0$ then $\alpha(P^*) < 0$. Otherwise, $\alpha(P) = 0$.*

Proof. P is diagonally dominant and has negative diagonal entries. Therefore by the Gershgorin Disc Theorem the real parts of all the eigenvalues are non-positive. Suppose $\gamma_k < 1, w_k^x > 0$, then strong connectivity of the opinion graph shows there is a path to all $i \in \{N+1, \dots, 2N\} \setminus \{N+k\}$ and the fact that $\beta_{ii} > 0, \forall i$ implies that there is a path to all $i \in \{1, \dots, N\}$. Then P is non singular by Lemma 3 and as the real parts of all the eigenvalues are non-positive, $\alpha(P) < 0$. To see that $\alpha(P) = 0$ when $\gamma_i = 1, \forall i$ consider the vector 1_{2N} . As $P1_{2N} = 0_{2N}$, 1_{2N} is an eigenvector with a corresponding zero eigenvalue. As the real parts of the eigenvalues of P are non-positive, $\alpha(P)$ must be zero. ■

In the following proof we consider an upper bound of (14):

$$P = \begin{bmatrix} -\bar{B} & \bar{B} \\ W & -(\mathcal{L}_o + W) \end{bmatrix},$$

We are now ready to prove Theorem 1.

Proof of Theorem 1. As the opinion graph is strongly connected and $w_i > 0, \forall i$ then the matrix P is irreducible and as such the eigenvalue 0 is simple [70]. By Lemma 5, there exists a positive diagonal matrix Q such that $P^T Q + QP$ is negative semidefinite. Consider the Lyapunov function $V(z) = z^T Qz$. Then by Lemma 6

$$\begin{aligned} \dot{V}(z) &= \dot{z}^T Qz + z^T Q\dot{z} \\ &\leq z^T (P^T Q + QP)z \leq 0. \end{aligned}$$

In what follows we will argue that $\dot{V}(z) < 0, \forall z \neq 0_{2N}$, allowing the use of $V(z)$ to show stability by Lyapunov's direct method. In the case that the upper bound $z^T (P^T Q + QP)z < 0, \dot{V}(z) < 0$ follows trivially. However when the upper bound $z^T (P^T Q + QP)z = 0$, more analysis is required to show that $\dot{V}(z) < 0$. Since Q is a positive diagonal matrix, $z^T (P^T Q + QP)z = 0$ is achieved when z equals the right eigenvector of P associated with zero, i.e. when $z \in \text{span}(1_{2N})$.

Consider the dynamics for x_i when $x_i = o_i \neq 0$ and $x_i = o_i \neq 1$ then by the condition on the adoption parameters

$$\begin{aligned} \dot{x}_i &= -\delta_i x_i (1 - o_i) + (1 - x_i) o_i \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j + \beta_{ii} \right) \\ &= -\delta_i x_i (1 - x_i) + (1 - x_i) x_i \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j + \beta_{ii} \right) \\ &= x_i (1 - x_i) \left(-\delta_i + \sum_{\mathcal{N}_i^A} \beta_{ij} x_j + \beta_{ii} \right) \\ &\leq x_i (1 - x_i) (-\delta_i + \Omega_i(\tau)) \\ &< 0. \end{aligned}$$

Considering the opinion dynamic, with the condition that $o_i = o_j, \forall i, j \in \{1, \dots, N\}$ because $z \in \text{span}(1_{2N})$ then

$$\begin{aligned} \dot{o}_i &= \sum_{j \in \mathcal{N}_i^O} (o_j - o_i) + w_i^x (\gamma_i x_i - o_i) \\ &\leq w_i^x (x_i - o_i) = 0. \end{aligned}$$

Then separating Q into $Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}$ we have that

$$\begin{aligned} \dot{V} &= x^T Q_1 \dot{x} + o^T Q_2 \dot{o} \\ &\leq x^T Q_1 \dot{x} < 0 \end{aligned}$$

where negativity holds as Q_1 is a positive diagonal matrix.

Having shown when $z^T (P^T Q + QP)z = 0$ that $\dot{V} < 0$, we now consider the two remaining cases, that $z \notin \text{span}(1_{2N})$ and $z = 0_{2N}$. In the considered domain, $z^T (P^T Q + QP)z < 0$ when $z \notin \text{span}(1_{2N})$, which implies that $\dot{V} < 0$. When $z = 0_{2N}$, the Lyapunov function satisfies $V(0_{2N}) = 0$ and $\dot{V}(0_{2N}) = 0$. Then the form of the Lyapunov function shows that $V(z) > 0, z \neq 0_{2N}$ on the considered domain. The above argument shows that $\dot{V}(z) < 0, z \neq 0_{2N}$. These together show the stability of 0_{2N} via Lyapunov's direct method. ■

Next we show the stability results of $z^* = 1_{2N}$.

Lemma 8. *The equilibrium point $z^* = 1_{2N}$ is locally stable if $\forall i, \Omega_i(1) > \delta_i, \gamma_i = 1$.*

Proof. As $\gamma_i = 1, \forall i, z^* = 1_{2N}$ is an equilibrium point. Consider the Jacobian matrix at the equilibrium point $z^* = 1_{2N}$

$$J(z^*) = \left[\begin{array}{c|c} \text{diag}(-\Omega_i(1)) & \text{diag}(\delta_i) \\ \hline W & -(\mathcal{L}_o + W) \end{array} \right].$$

Similarly to the case of the equilibrium at 0_{2N} , the condition $\Omega_i(1) > \delta_i, \forall i$ and Assumptions 2 and 4 show that $J(z^*)$ is WCDD. Since the diagonal elements are negative,

the Jacobian is Hurwitz, by the Gershgorin Disc Theorem, which implies $z = 1_{2N}$ is a locally stable equilibrium. ■

Proof of Theorem 2. To show asymptotic stability of 1_{2N} , consider the change of variables $\hat{x}_i = 1 - x_i$ and $\hat{o}_i = 1 - o_i$. Then $\dot{\hat{x}}_i = -\dot{x}_i$ and $\dot{\hat{o}}_i = -\dot{o}_i$. It follows that:

$$\begin{aligned}\dot{\hat{x}}_i &= \delta_i x_i (1 - o_i) - (1 - x_i) o_i \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j + \beta_{ii} \right) \\ &= \delta_i (1 - \hat{x}_i) \hat{o}_i - \hat{x}_i (1 - \hat{o}_i) \left(\sum_{\mathcal{N}_i^A} \beta_{ij} (1 - \hat{x}_j) + \beta_{ii} \right)\end{aligned}$$

and

$$\begin{aligned}\dot{\hat{o}}_i &= \sum_{\mathcal{N}_i^O} (o_i - o_j) + w_i^x (o_i - \gamma_i x_i) \\ &= \sum_{\mathcal{N}_i^O} (\hat{o}_j - \hat{o}_i) + w_i^x (\hat{x}_i - \hat{o}_i)\end{aligned}$$

as $\gamma_i = 1, \forall i$. Consider the \hat{x}_i dynamic:

$$\begin{aligned}\dot{\hat{x}}_i &= \delta_i (1 - \hat{x}_i) \hat{o}_i - \hat{x}_i (1 - \hat{o}_i) \left(\sum_{\mathcal{N}_i^A} \beta_{ij} (1 - \hat{x}_j) + \beta_{ii} \right) \\ &\leq \delta_i (1 - \hat{x}_i) \hat{o}_i - \hat{x}_i (1 - \hat{o}_i) (\Omega_i(\tau)) \\ &< \delta_i (1 - \hat{x}_i) \hat{o}_i - \hat{x}_i (1 - \hat{o}_i) (\delta_i) \\ &= \delta_i (\hat{o}_i - \hat{x}_i).\end{aligned}$$

Then the matrix

$$\hat{P} = \begin{bmatrix} -D & D \\ W & -(\mathcal{L}_o + W) \end{bmatrix}, \quad (15)$$

where $D = \text{diag}(\delta_i)$, satisfies $\dot{\hat{z}} \leq \hat{P}\hat{z}$.

Similar to the proof of Theorem 1, if the opinion graph is strongly connected and $w_i^x > 0, \forall i$ then \hat{P} is irreducible and by Lemma 5, there exists a \hat{Q} that renders $\hat{P}'\hat{Q} + \hat{Q}\hat{P}$ negative semi-definite. Then as in Theorem 1, one can use $\hat{z}^T \hat{Q} \hat{z}$ as a Lyapunov function to show stability of 1_{2N} . ■

The characterization of the unstable equilibrium follows.

Proof of Lemma 2. Consider the dynamic in x_i at the point z^* under the assumption that $\delta_i = \sum_{\mathcal{N}_i^A} \beta_{ij} x_j^* + \beta_{ii}$:

$$\begin{aligned}\dot{x}_i &= -\delta_i (1 - o_i^*) x_i^* + (1 - x_i^*) o_i^* \left(\sum_{\mathcal{N}_i^A} \beta_{ij} x_j^* + \beta_{ii} \right) \\ &= -\delta_i (1 - o_i^*) x_i^* + (1 - x_i^*) o_i^* \delta_i \\ &= \delta_i (o_i^* - x_i^*)\end{aligned}$$

then $\dot{x}_i = 0$ if $o_i^* = x_i^*$. Substituting in the other conditions:

$$\begin{aligned}\dot{o}_i &= \sum_{j \in \mathcal{N}_i^O} (o_j^* - o_i^*) + (\gamma_i x_i^* - o_i^*) \\ &= \sum_{j \in \mathcal{N}_i^O} (x_j^* - x_i^*) + (x_i^* - x_i^*) \\ &= \sum_{j \in \mathcal{N}_i^O} (x_j^* - x_i^*)\end{aligned}$$

Then if $\sum_{j \in \mathcal{N}_i^O} (x_j^* - x_i^*) = 0, \dot{o}_i = 0$. If these conditions hold for all i then z^* is an equilibrium point. To see that the

solution is unique consider the matrix equation which the equilibrium z^* must satisfy

$$\begin{bmatrix} B & 0 \\ I & -I \\ -\mathcal{L}_o & 0 \end{bmatrix} z^* = \begin{bmatrix} D - \bar{B} \\ 0 \\ 0 \end{bmatrix}.$$

By Assumption 1, it holds that $\text{rank} \left(\begin{bmatrix} B & 0 \\ I & -I \\ -\mathcal{L}_o & 0 \end{bmatrix} \right) = 2N$.

By the Rouché-Capelli Theorem, a solution exists and is unique [68]. ■

Proof of Theorem 3. Consider the Jacobian at the equilibrium point z^* , the properties of which are described in Lemma 2. The derivatives of the adoption dynamic at z^* follow

$$\begin{aligned}\frac{\partial f_i}{\partial x_i} &= -\delta_i \\ \frac{\partial f_i}{\partial x_j} &= \begin{cases} (1 - x_i^*) x_i^* \beta_{ij} & \text{if } j \in \mathcal{N}_i^A, j \neq i \\ 0 & \text{if } j \notin \mathcal{N}_i^A \cup \{i\} \end{cases} \\ \frac{\partial f_i}{\partial o_i} &= \delta_i \\ \frac{\partial f_i}{\partial o_j} &= 0 \quad \forall j \neq i.\end{aligned}$$

The Jacobian can be written as

$$J(z^*) = \left[\begin{array}{c|c} \frac{\partial f}{\partial x} & D \\ \hline W & -(\mathcal{L}_o + W) \end{array} \right]$$

where $D = \text{diag}(\delta_i)$. If the opinion graph \mathcal{G}_O is strongly connected, the Jacobian is irreducible, which allows the application of Lemma 4. Consider a vector y of the form $y = [\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_N \quad 1_N^T]^T$ where $\alpha_i = 1 + \epsilon_i$ and $0 < \epsilon_i < \frac{\sum_{\mathcal{N}_i^A} \beta_{ij} (1 - x_i^*) x_i^*}{\delta_i}$. Then consider the matrix product $J(z^*)y$. The first N entries of $J(z^*)y$ follow:

$$\begin{aligned}& -\delta_i \alpha_i + \alpha_i \left(\sum_{\mathcal{N}_i^A} \beta_{ij} (1 - x_i^*) x_i^* \right) + \delta_i \\ & > -\delta_i \alpha_i + \left(\sum_{\mathcal{N}_i^A} \beta_{ij} (1 - x_i^*) x_i^* \right) + \delta_i \\ & > -\delta_i \epsilon_i + \left(\sum_{\mathcal{N}_i^A} \beta_{ij} (1 - x_i^*) x_i^* \right) \\ & > - \left(\sum_{\mathcal{N}_i^A} \beta_{ij} (1 - x_i^*) x_i^* \right) + \left(\sum_{\mathcal{N}_i^A} \beta_{ij} (1 - x_i^*) x_i^* \right) \\ & = 0.\end{aligned}$$

The last N entries of $J(z^*)y$ follow

$$\begin{aligned}\alpha_i w_i^x - d_i^O - w_i^x + \sum_{\mathcal{N}_i^O} 1 &= (\alpha_i - 1) w_i^x \\ &= \epsilon_i w_i^x \\ &> 0.\end{aligned}$$

As y is elements-wise positive and the resulting vector $J(z^*)y$ is element-wise positive, by Lemma 4, $\alpha(J(z^*)) > 0$ and the equilibrium point is unstable. ■

Finally, we show stability of the adoption dynamic coupled with the bounded confidence opinion dynamic model shown in (12). The non-positivity of the eigenvalues of the matrix P from (14) and \hat{P} from (15), which are used to characterize the stability of the equilibria of the coupled adoption behavior, does not depend on the structure of the opinion graph. As such the results for the stability of the equilibria of the coupled adoption model can be extended to the bounded confidence opinion dynamic model if the matrix associated with the upper bound is shown to be negative semidefinite. Proving asymptotic stability requires the following definitions from the study of switched systems, which follow [71]. Consider a family of systems, with some index set \mathcal{P} ,

$$\dot{x} = f_p(x) \quad p \in \mathcal{P} \quad (16)$$

which has a switching signal $\sigma(t) : [0, \infty) \rightarrow \mathcal{P}$ which determines the switches between systems. This gives rise to a switched system,

$$\dot{x} = f_\sigma(x). \quad (17)$$

Definition 3. A switched system is uniformly asymptotically stable if it is asymptotically stable for all switching signals.

Definition 4. A positive definite \mathcal{C}^1 function V is a common Lyapunov function for the family of systems in (16) if there is a positive definite continuous function W such that

$$\frac{\partial V}{\partial t} f_p(x) \leq -W(x) \quad \forall x \neq 0, \quad \forall p \in \mathcal{P}$$

or equivalently if \mathcal{P} is compact and

$$\frac{\partial V}{\partial t} f_p(x) < 0 \quad \forall x \neq 0, \quad \forall p \in \mathcal{P}.$$

Lemma 9 (Theorem 2.1 from [71]). *If all systems in the family in (16) share a common Lyapunov function, then the switched system in (17) is uniformly asymptotically stable.*

Proof of Theorem 4. Consider the finite collection of opinion graph topologies $\hat{\mathcal{G}}_o = \{\mathcal{G}_o^1, \mathcal{G}_o^2, \dots, \mathcal{G}_o^s\}$ which the bounded confidence model can switch between. The original opinion graph $\mathcal{G}_o \in \hat{\mathcal{G}}_o$, as well as the empty opinion graph $\mathcal{G}_o^\emptyset \in \hat{\mathcal{G}}_o$. Consider the graph \mathcal{G}_o^i which consists of k connected subgraphs. Then under the opinion dynamic on \mathcal{G}_o^i , and if $\delta_i > \Omega_i(1)$, $\forall i$, the dynamics follow:

$$\dot{z} \leq P_i z$$

where

$$P_i = \begin{bmatrix} -\bar{B} & & & \bar{B} \\ & \left(\begin{bmatrix} \mathcal{L}_o^1 & 0 & \dots & 0 \\ 0 & \mathcal{L}_o^2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & \mathcal{L}_o^k \end{bmatrix} + W \right) & & \\ W & & & \end{bmatrix},$$

though a permutation may be required to put the opinion dynamics into this form. The matrix P_i is negative semidefinite as it is diagonally dominant with negative diagonal elements and symmetric by assumption. Under \mathcal{G}_o^\emptyset and if $\delta_i > \Omega_i(1) \forall i$ the dynamics follow

$$\dot{z} \leq \begin{bmatrix} -\bar{B} & \bar{B} \\ W & -W \end{bmatrix} z = P_\emptyset z.$$

The matrix P_\emptyset is also negative semidefinite as it is diagonally dominant with negative diagonal elements and symmetric by assumption. Then under any opinion graph $\mathcal{G}^j \in \hat{\mathcal{G}}_o$ the function $V(z) = \frac{1}{2} z^T z$ satisfies

$$\begin{aligned} \dot{V} &\leq z^T P_j z \\ &\leq 0. \end{aligned}$$

Unlike the case of Theorem 1, there are now multiple eigenvectors associated with the zero eigenvalue, specifically one for each connected component of the subgraph. In the extreme case of the empty opinion graph \mathcal{G}_o^\emptyset , if node i satisfies $x_i = o_i = 1$ then it will stay there indefinitely independent of the behavior of the other nodes in the system as this is an equilibrium point for x_i and o_i . Now if $x_i = o_i = c$ for $0 < c < 1$ one can use a similar argument to Theorem 1 to show that the chosen Lyapunov function has $\dot{V} < 0$.

Therefore, the domain over which stability is considered has been modified to $[0, 1)^{2N}$ to prevent such occurrences. This domain excludes the eigenvectors of P_j associated with the zero eigenvalue that are also equilibrium points of the system. Therefore on the domain $[0, 1)^{2N}$, $\dot{V}(z) < 0$, $\forall z \neq 0_{2N}$. As the set of possible graph topologies is finite, $V(x) = \frac{1}{2} z^T z$ serves as a common Lyapunov function by Definition 4 and can be used to show that the system is uniformly asymptotically stable by Lemma 9. ■

The theorem for the stability of the hit equilibrium point $z^* = 1_{2N}$ is presented without proof as the proof follows similarly to the proof of Theorem 4.