Stability of Leaderless Resource Consumption Networks

Sebastian F. Ruf[1], Matthew T. Hale[2], Talha Manzoor[3], Abubakr Muhammad[4]

Abstract— **In this paper, we study the global stability properties of a multi-agent model of natural resource consumption, which balances ecological and social network components in determining the consumption behavior of a group of agents. Recently, it was shown that if the social network component of the model is leaderless, a condition that ensures that no single node has a greater social influence than any other node on the dynamics of the resource consumption, then the behavior of a group of agents can be treated in aggregate. This aggregation facilitates the application of this model to large scale networks, however it is as yet poorly understood. This paper shows that any network structure can be made leaderless by the social preferences of the agents. It is also shown that if the social network is leaderless, a mild bound on agents' environmental concern is sufficient for global asymptotic stability to a positive consumption value; indicating that appropriately configured networks can consume without depleting the resource. The behavior of these leaderless resource consumption networks is discussed via simulation.**

I. INTRODUCTION

In the face of an ever-changing natural climate, understanding the behavior of renewable natural resources and the impact of human consumption on those resources is important for ensuring long term resource consumption [1], [2]. Modeling of natural resources offers valuable insights into the effect of various system components, such as network structure or the social preferences of the agents, on consumption. Of particular interest is the equilibrium behavior of these models, as equilibria can help describe the long term sustainability of natural resources [3]. The discussion of long term system behavior must be preceded by an understanding of the stability properties of the system.

This paper focuses on the study of an agent-based model of natural resource consumption previously introduced and studied in [4]–[6]. This model captures insights from the social sciences on the consumption behavior of humans in a form that can be analyzed mathematically. Past work on this model has sought to understand the behavior of the model and has considered stability of this model in the two agent case. This paper extends the consideration of stability to consider n agents interacting over a leaderless network.

The overall system consists of an ecological sub-model, describing the resource dynamics, and a social sub-model, describing the consumption behavior of the agents. The ecological sub-model is based on the Gordon Schaefer model, which represents a class of well studied dynamic processes in population biology, ecological economics and other related disciplines [7], [8]. The stability of the Gordon-Schaefer model, as well as similar logistic growth models, has been studied extensively in isolation from the social processes which drive human consumption behavior [9]–[12].

The social sub-model describes the process through which agents decide to change their resource consumption. This process is influenced both by the state of the resource as well as the consumption of neighboring agents. The influence of the agents on each others' consumption is similar to how agents influence each other in mathematical models of opinion formation [13] and consensus in cooperative multi-agent systems [14]. The dependence of the agents' resource consumption on the state of the resource appears as an exogenous factor or time-varying bias in the overall dynamics (see [15], [16] for similar models). An important component of the social process is the underlying social network structure, which influences the ability of a community to successfully manage its natural resources [17], [18].

This paper studies the behavior of a leaderless consumption network; in these networks no one agent will drive the social network component of the model more than any other agent in the network. This assumption allows an aggregation of individual state nodes [5], facilitating an understanding of the system-level behavior. Discussing a consuming population in aggregate is a common tool for the study of resource consumer social networks [19] and actions taken to change behavior often happen at the community level [18]. Therefore understanding leaderless networks is an important step towards the application of this model. As will be shown, leaderless networks can describe a wide array of network topologies and exhibit a rich class of stable system behaviors on the individual scale.

The rest of the paper is organized as follows. Section II introduces the consumption model and discusses its properties. Section III discusses the leaderless condition and presents a Lyapunov based proof for global stability of the system. In Section IV the behavior of leaderless systems is studied in simulation, with a discussion and conclusion in Section V.

II. SYSTEM DYNAMICS

This section presents the dynamics governing the resource quantity and consumer behavior in the coupled socio-ecological system. We first discuss each sub-model and then give the aggregate leaderless consumption model.

[1] S.F. Ruf is with the Center for Complex Networks Research, Northeastern University, Boston, MA. Email:s.ruf@northeastern.edu

[2] M.T. Hale is with the Department of Mechanical and Aerospace Engineering, University of Florida, Gainesville, FL. Email: matthewhale@ufl.edu

[3] T. Manzoor is with the Department of Electrical Engineering, Namal College, Mianwali, Pakistan. Email: talha@namal.edu.pk

[4] A. Muhammad is with the Department of Electrical Engineering, Lahore University of Management Sciences (LUMS), Lahore, Pakistan. Email: abubakr@lums.edu.pk

A. The Ecological Sub-model

The ecological component of the system is assumed to consist of a single renewable resource with quantity at time τ represented by $R(\tau)$. In the absence of consumption, the resource grows at intrinsic growth rate r and saturates at carrying capacity $\mathrm{R}_{\max}$. The resource is connected to a consuming population consisting of n individuals. Each individual can harvest the resource by exerting consumption effort $e_i(\tau)$, where $i \in \{1, \ldots, \mathrm{n}\}$ represents a single consumer. The resource dynamics are assumed to follow the standard Gordon-Schaefer model [8] with catch coefficient equal to one, which is given as

$$\frac{dR(\tau)}{d\tau} = \mathrm{r}R(\tau)\left(1 - \frac{R(\tau)}{\mathrm{R}_{\max}}\right) - R(\tau)\sum_{i=1}^{\mathrm{n}} e_i(\tau). \qquad (1)$$

Note that Equation 1 implies that the effort $e_i(\tau)$ represents the fraction of i's harvest w.r.t the resource $R(\tau)$, while the actual harvest is given by the product $e_i(\tau)\,R(\tau)$ (the interested reader is directed to [4] for detailed discussions on the origin of the model).

B. The Social Sub-model

The social sub-model is based on Festinger's theory of social comparison processes [20], which postulates that human beings evaluate their decisions by reflecting on both objective and social information. In the context of natural resource consumption, objective information corresponds to the state of the resource and social information corresponds to the consumption of other socially connected individuals [21]. To balance between objective and social information, the change in consumption effort of the agent is given as a weighted sum of both ecological and social factors.

The ecological factor for consumer i is given by $R(\tau) - \mathrm{R}_i$, where $\mathrm{R}_i \in \mathbb{R}$ represents the perceived scarcity threshold of i, below which agent i considers the resource to be scarce, and above which she considers it to be abundant. The ecological factor is weighed by $\mathrm{a}_i \in (0, \infty)$, which represents the set of factors to which agent i attributes the state of the natural resource. An ecological attribution $\mathrm{a}_i \to 0$ represents a consumer that attributes the state of the resource entirely to the actions of the consuming society (including the agent itself), while increasing values of a_i correspond to the individual attributing the current state of the resource to natural causes (droughts, wildfires, heavy rain, etc).

The ecological factor is balanced by a social component, given by $\sum_{i=1}^{\mathrm{n}} \omega_{ij}(e_j(\tau) - e_i(\tau))$, which is the difference between i's consumption and that of the other socially connected consumers in the population. The graph connectivity is captured by $\omega_{ij} \geq 0$ which is the strength of the social tie directed from j to i. We assume that $\sum_{j=1}^{\mathrm{n}} \omega_{ij} = 1$ and $\omega_{ii} = 0\,\forall\, i \in \{1, \ldots, \mathrm{n}\}$. For convenience, we assume a single connected component; however no presented results rely on that fact. The social factor is weighed by $\mathrm{s}_i \in (0, \infty)$, the social-value orientation of i. Social-value orientations $\mathrm{s}_i \to 0$ represent extremely non-cooperative individuals, which will ignore the actions of their network neighbors. Conversely, increasing values of s_i correspond to increasingly cooperative individuals.

Combining the social and ecological component gives the dynamics of the consumption effort for consumer i as

$$\frac{de_i(\tau)}{d\tau} = \mathrm{a}_i(R(\tau) - \mathrm{R}_i) + \mathrm{s}_i \sum_{j=1}^{\mathrm{n}} \omega_{ij}(e_j(\tau) - e_i(\tau)), \qquad (2)$$

where the ecological and social factors have been weighed in accordance with findings in social psychological research on consumer behavior [4]. In particular, individuals that attribute blame to natural causes tend to give more importance to ecological information and vice versa. Similarly, cooperative individuals are more concerned with maximizing equality in consumption than non-cooperative ones, and as such will be further influenced by the social factor.

It is important to note that (2) captures a general notion of effort which is based in the ecology literature [8]. As such, the effort e_i can take on negative values. Negative effort relates to contributing to the sustenance of the resource (as opposed to harvesting from it). The interested reader is directed to [4] and included references for physical interpretations of this phenomenon and its implications on the overal scope of the model.

C. Non-dimensionalized Socio-Ecological System

In order to reduce the dimensionality of the parameter space, the system given by Eq. (1) and (2) is non-dimensionalized, which also has an added benefit of allowing comparison between system parameters. The dynamics of the non-dimensionalized state of the resource $x = \frac{R(\tau)}{\mathrm{R}_{\max}}$ and the non-dimensionalized consumption $y_i = \frac{e_i}{\mathrm{r}}$ are given as follows

$$\begin{aligned} \dot{x} &= (1-x)x - x\sum_{i=1}^{\mathrm{n}} y_i, \\ \dot{y}_i &= \mathrm{b}_i\left((1-\nu_i)(x - \rho_i) + \nu_i \sum_{j=1}^{\mathrm{n}} \omega_{ij}(y_j - y_i)\right), \end{aligned} \qquad (3)$$

where $i \in \{1, \ldots, \mathrm{n}\}$,

$$\mathrm{b}_i = \frac{\mathrm{a}_i\mathrm{R}_{\max} + \mathrm{r}\mathrm{s}_i}{\mathrm{r}^2}, \quad \nu_i = \frac{\mathrm{r}\mathrm{s}_i}{\mathrm{a}_i\mathrm{R}_{\max} + \mathrm{r}\mathrm{s}_i},$$

and the derivatives $\dot{x}$ and $\dot{y}_i$ are taken with respect to the non-dimensional time $t = \mathrm{r}\tau$. It can be shown using standard approaches, e.g., [22, Theorem 3.1], that a unique solution to this system exists for all time. Below we will also use the notation $\alpha_i = 1 - \nu_i$. The non-dimensionalized threshold $\rho_i = \frac{\mathrm{R}_i}{\mathrm{R}_{\max}}$ is called the environmentalism of i. The parameter b_i is the sensitivity of i, which represents $i's$ openness to change in her consumption. The final parameter, ν_i, is called the socio-ecological relevance of i and represents the importance that i gives to social information relative to ecological information in the process of changing consumption behavior.

D. Influence and Leadership

The consumption of i is influenced by the consumption of all other agents that are socially connected to her. This notion of connectivity is captured in Eq. (3) via the parameters ω_{ij}, which denote the strength of the social tie directed from j to i. If $\omega_{ij} = 0$ this implies that there is no social link from j to i, allowing the collection of ω_{ij}'s to specify the topology of the underlying social network. The aggregate influence of the rest of the agents on i's consumption is given by $\sum_{j=1}^{\mathrm{n}} \mathrm{b}_i \nu_i \omega_{ij}$ and is called the in-influence of i. The aggregate influence that i exerts on the other agents in the network is given by $\sum_{j=1}^{\mathrm{n}} \mathrm{b}_j \nu_j \omega_{ji}$ and is called the out-influence of i. The difference between the out-influence and the in-influence is called the net-influence of i and determines the role of i in the network as a leader (positive net-influence), a follower (negative net-influence) or neutral (zero net-influence). In this paper, we consider cases in which all agents in the network are neutral, i.e. the network is leaderless.

III. Global Asymptotic Stability of Leaderless Networks

In this section, two assumptions on the network and parameters are introduced before transforming the non-dimensionalized dynamics in Eq. (3) into a form more amenable to stability analysis. Following this, global asymptotic stability to an equilibrium point is shown for the non-dimensionalized dynamics in Equation (3).

A. Leaderless Networks

The section considers the first of two assumptions on the system under consideration.

Assumption 1: The network is leaderless, i.e.,

$$\sum_{j=1}^{\mathrm{n}} \left(\omega_{ij}\mathrm{b}_i\nu_i - \omega_{ji}\mathrm{b}_j\nu_j\right) = 0$$

for all $i \in \{1, \dots, \mathrm{n}\}$. △

See [5] for a further discussion of the implications of this assumption. The assumption implies that the net-influence (as defined in Section II-D) of each individual is zero i.e., there are no leaders or followers in the network. As seen later, this implies that in aggregate the social network does not have an effect on resource consumption, though the social network still has a contribution to individual consumption behavior. While this may appear to be a strong assumption on the network, we demonstrate that it is possible to have any network topology be leaderless based on the social value orientations $\mathrm{s}_i = \mathrm{r}\mathrm{b}_i\nu_i$.

Lemma 1: For any set of network weights ω_{ij}, there exists a set of social value orientations $\{\mathrm{s}_1, \mathrm{s}_2, \dots, \mathrm{s}_n\}$ that renders the network leaderless.

Proof: The leaderless condition can be expressed as $\sum_{j=1}^{\mathrm{n}} \left(\omega_{ij}\mathrm{s}_i - \omega_{ji}\mathrm{s}_j\right) = 0,\ \forall i$. Consider the matrix of edge weights,

$$W = \begin{bmatrix} -\left(\sum_{j=2}^{n} \omega_{1j}\right) & \omega_{21} & \dots & \omega_{n1} \\ \omega_{12} & -\left(\sum_{\substack{j=1 \\ j\neq 2}}^{n} \omega_{2j}\right) & \dots & \omega_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{1n} & \omega_{2n} & \dots & -\left(\sum_{j=1}^{n-1} \omega_{nj}\right) \end{bmatrix}.$$

If the vector of social value orientations $s \in \mathrm{null}(W)$ then the system is leaderless. Notice that the matrix W^T has rows that sum to 0, i.e. the vector 1_n is an eigenvector with eigenvalue 0. As W and W^T have the same eigenvalues [23], W also has an eigenvalue at 0. Further, from the structure of W, any element of the null space must have all nonzero elements with the same sign. To see this, suppose $e \in \mathrm{null}(W)$, with $e_i < 0$ while $e_j > 0,\ \forall j \neq i$. Then $W(i,:)e > 0$ and $e \notin \mathrm{null}(W)$. Therefore for a given W there exists an element wise positive $\mathrm{s} \in \mathrm{null}(W)$ and if the agents have those social value orientations the network is leaderless. ■

Lemma 1 shows that any graph, including those commonly found in complex networks such as scale free [24] and small world [25] networks, can be rendered leaderless by the appropriate social value orientation. As such, Assumption 1 is widely applicable. The behavior of the resource dynamic over a leaderless network will be studied in Section IV, after the stability of the system has been established.

B. Dynamics

This section considers a second assumption as well as its implication for the system level dynamics. In what follows, the following new state variables will be considered

$$z = \log x \quad \text{and} \quad u = \sum_{i=1}^{\mathrm{n}} y_i.$$

Further the following assumption, which bounds the maximum possible value of ρ_i for each agent, will be enforced.

Assumption 2: For all $i \in \{1, \dots, \mathrm{n}\}$, $\rho_i \in (0, 2)$. △

Because ρ_i is the normalized value of R_i, Assumption 2 implies that $\mathrm{R}_i \in (0, 2\mathrm{R}_{\text{max}})$. This is a rather weak assumption as few agents are expected to have $\mathrm{R}_i > \mathrm{R}_{\text{max}}$, i.e. a scarcity threshold *larger* than the resource carrying capacity.

With these assumptions in place, the transformed network level dynamics will be derived. Computing the time derivative of z gives

$$\dot{z} = \frac{\dot{x}}{x} = 1 - x - \sum_{i=1}^{\mathrm{n}} y_i = 1 - e^z - u.$$

Differentiating u with respect to time and expanding gives

$$\dot{u} = \sum_{i=1}^{\mathrm{n}} \dot{y}_i,$$

$$\dot{u} = \sum_{i=1}^{\mathrm{n}} \left(\mathrm{b}_i\alpha_i x - \mathrm{b}_i\alpha_i\rho_i\right) - \sum_{i=1}^{\mathrm{n}}\sum_{j=1}^{\mathrm{n}} \mathrm{b}_i\nu_i\omega_{ij}(y_i - y_j)$$

$$= \sum_{i=1}^{\mathrm{n}} \left(\mathrm{b}_i\alpha_i e^z - \mathrm{b}_i\alpha_i\rho_i\right)$$

$$
\begin{aligned}
&- \sum_{i=1}^{\mathrm{n}} \left[\sum_{j=1}^{\mathrm{n}} \left(\omega_{ij} \mathrm{b}_i \nu_i - \omega_{ji} \mathrm{b}_j \nu_j \right) \right] y_i \\
&= \sum_{i=1}^{\mathrm{n}} \mathrm{b}_i \alpha_i e^z - \sum_{i=1}^{\mathrm{n}} \mathrm{b}_i \alpha_i - \sum_{i=1}^{\mathrm{n}} \mathrm{b}_i \alpha_i (\rho_i - 1),
\end{aligned}
$$

where the sum containing y_i has vanished due to Assumption 1. Continuing,

$$
\begin{aligned}
\dot{u} &= \left[\sum_{i=1}^{\mathrm{n}} \mathrm{b}_i \alpha_i \right] (e^z - 1) - \sum_{i=1}^{\mathrm{n}} \mathrm{b}_i \alpha_i (\rho_i - 1) \\
&= \mathrm{K}_1 (e^z - 1) - \mathrm{K}_2,
\end{aligned}
$$

where

$$
\mathrm{K}_1 := \sum_{i=1}^{\mathrm{n}} \mathrm{b}_i \alpha_i \text{ and } \mathrm{K}_2 := \sum_{i=1}^{\mathrm{n}} \mathrm{b}_i \alpha_i (\rho_i - 1).
$$

From these definitions, K_1 is manifestly positive because it is a sum of positive terms. Under Assumption 2, it also follows that

$$
\begin{aligned}
|\mathrm{K}_2| &= \left| \sum_{i=1}^{\mathrm{n}} \mathrm{b}_i \alpha_i (\rho_i - 1) \right| \leq \max_{i \in \{1, \ldots, \mathrm{n}\}} |\rho_i - 1| \sum_{i=1}^{\mathrm{n}} \mathrm{b}_i \alpha_i \\
&= \mathrm{K}_1 \max_{i \in \{1, \ldots, n\}} |\rho_i - 1| \leq \mathrm{K}_1,
\end{aligned}
$$

where the last inequality follows from Assumption 2.

The (z, u) dynamics thus take the form

$$
\begin{aligned}
\dot{z} &= 1 - e^z - u \\
\dot{u} &= \mathrm{K}_1 (e^z - 1) - \mathrm{K}_2.
\end{aligned}
$$

Next, the equilibrium of the (z, u) system is computed to translate the equilibrium of the system to the origin.

C. Equilibrium

The following lemma provides the uniqueness and value of the (z, u) system's equilibrium point.

Lemma 2: The (z, u) system has a unique equilibrium point located at

$$
\begin{aligned}
\mathrm{z}_0 &= \log \left(\frac{\mathrm{K}_2}{\mathrm{K}_1} + 1 \right) \\
\mathrm{u}_0 &= -\frac{\mathrm{K}_2}{\mathrm{K}_1}.
\end{aligned}
$$

Proof: Setting $\dot{u} = 0$ we find

$$
\dot{u} = \mathrm{K}_1 (e^z - 1) - \mathrm{K}_2 = 0,
$$

which immediately provides

$$
\mathrm{z}_0 = \log(\mathrm{K}_2 / \mathrm{K}_1 + 1).
$$

Setting $\dot{z} = 0$ gives

$$
\dot{z} = 1 - e^z - u = 0,
$$

where setting $z = \mathrm{z}_0$ results in

$$
\dot{z} = 1 - \left(\frac{\mathrm{K}_2}{\mathrm{K}_1} + 1 \right) - u = 0.
$$

Solving for u_0 then provides

$$
\mathrm{u}_0 = -\frac{\mathrm{K}_2}{\mathrm{K}_1}.
$$

■

By Lemma 2, the equilibrium value of the resource, R_0, is

$$
\mathrm{R}_0 = \left(\frac{\mathrm{K}_2}{\mathrm{K}_1} + 1 \right) \mathrm{R}_{\text{max}} = \frac{\sum_{i=1}^{n} \mathrm{a}_i \mathrm{R}_i}{\sum_{j=1}^{n} \mathrm{a}_j}.
$$

Here, the value of R_0 is always positive as $R_i \in (0, 2\mathrm{R}_{\text{max}}), \forall i$.

Having computed the equilibrium point of the system, we define a coordinate shift by

$$
v = z - \mathrm{z}_0, \quad w = u - \mathrm{u}_0,
$$

resulting in the dynamics

$$
\begin{aligned}
\dot{v} = \dot{z} &= 1 - e^z - u = 1 - e^{v + \mathrm{z}_0} - (w + \mathrm{u}_0), \\
&= -e^v e^{\mathrm{z}_0} - w + 1 + \frac{\mathrm{K}_2}{\mathrm{K}_1} = -e^v e^{\mathrm{z}_0} - w + e^{\mathrm{z}_0}, \\
&= -e^{\mathrm{z}_0} (e^v - 1) - w,
\end{aligned}
$$

where we have used $e^{\mathrm{z}_0} = \mathrm{K}_2 / \mathrm{K}_1 + 1$.

For w, the dynamics are governed by

$$
\begin{aligned}
\dot{w} = \dot{u} &= \mathrm{K}_1 (e^z - 1) - \mathrm{K}_2 \\
&= \mathrm{K}_1 e^{v + \mathrm{z}_0} - \mathrm{K}_1 - \mathrm{K}_2 \\
&= \mathrm{K}_1 e^v \left(1 + \frac{\mathrm{K}_2}{\mathrm{K}_1} \right) - \mathrm{K}_1 - \mathrm{K}_2 \\
&= \mathrm{K}_1 e^v + \mathrm{K}_2 e^v - \mathrm{K}_1 - \mathrm{K}_2 \\
&= (\mathrm{K}_1 + \mathrm{K}_2)(e^v - 1).
\end{aligned}
$$

The final system dynamics to be analyzed are

$$
\begin{aligned}
\dot{v} &= -e^{\mathrm{z}_0} (e^v - 1) - w \\
\dot{w} &= (\mathrm{K}_1 + \mathrm{K}_2)(e^v - 1),
\end{aligned}
\tag{5}
$$

whose unique equilibrium point is the origin.

D. Global Stability

The following theorem demonstrates asymptotic stability of trajectories of the system in Equation (5) to the origin.

Theorem 1: Under Assumptions 1 and 2 the origin is globally asymptotically stable in Equation 5.

Proof: Consider the Lyapunov function

$$
V(v, w) = e^v - v - 1 + \frac{(\mathrm{K}_1 + \mathrm{K}_2)^{-1}}{2} w^2,
$$

which is positive definite, satisfies $V(0, 0) = 0$, and is radially unbounded. Differentiating V with respect to time,

$$
\begin{aligned}
\dot{V} &= e^v \dot{v} - \dot{v} + (\mathrm{K}_1 + \mathrm{K}_2)^{-1} w \dot{w} \\
&= e^v (-e^{\mathrm{z}_0} (e^v - 1) - w) + e^{\mathrm{z}_0} (e^v - 1) \\
&\quad + w + w(e^v - 1) \\
&= -e^{\mathrm{z}_0} e^v (e^v - 1) - e^v w + e^{\mathrm{z}_0} (e^v - 1) \\
&\quad + w + w(e^v - 1) \\
&= -e^{\mathrm{z}_0} (e^v - 1)^2 + w(e^v - 1) - w(e^v - 1) \\
&= -e^{\mathrm{z}_0} (e^v - 1)^2 \leq 0.
\end{aligned}
$$

Here, LaSalle's invariance principle can be used to prove global asymptotic stability of $(0, 0)$ by showing that the set

$V_0 = \{(v,w) \mid \dot{V}(v,w) = 0\}$ contains only the trivial trajectory $\big(v(t), w(t)\big) \equiv (0,0)$ [22].

From above, observe that $\dot{V}(v,w) = 0$ only for trajectories of the form $(0,w)$. Plugging this into the system dynamics in Equation (5) gives

$$\dot{v} = -w, \quad \dot{w} = 0$$

for such trajectories. Then the only invariant trajectory in V_0 has $w \equiv 0$ because $\dot{v} = 0$ must hold to ensure that the system remains in V_0. ■

IV. Simulations

This section considers the behavior of leaderless network topologies in simulation, focusing specifically on the case of the star graph. The star graph is of central importance to the study of complex networks [26]. The star graph also has a node, the center of the star, that might be expected to be the leader of a social network. Despite this, there are many leaderless social networks that can evolve over the star graph. Three leaderless networks on the same star topology are shown in Figure 1. Note that as more influence is given to a node, the node tends to be more cooperative for the network to remain leaderless.

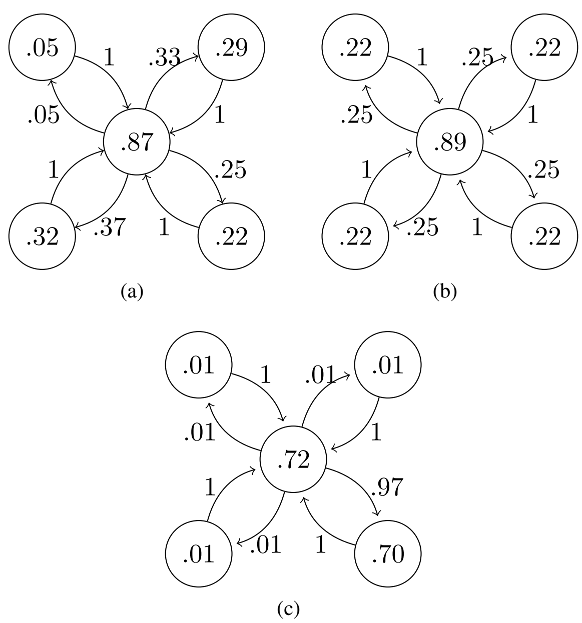

Fig. 1: 3 Leader-Less Star Graphs for (1a) random weights, (1b) uniform weights, (1c) skewed weights. Each edge is labeled with its weight and each node is labeled with its social value orientation.

The resource is assumed to have a carrying capacity $\mathrm{R}_{\text{max}} = 1$, a growth rate $\mathrm{r} = 1$, and a random initial condition that was fixed across simulations. The network was run with an attribution vector a and a set of scarcity thresholds R:

$$\mathrm{a} = \begin{bmatrix} 0.4340 \\ 0.2046 \\ 0.1891 \\ 0.6935 \\ 0.2108 \end{bmatrix}, \quad \mathrm{R} = \begin{bmatrix} 0.2262 \\ 0.4788 \\ 0.4582 \\ 1.1745 \\ 0.8483 \end{bmatrix}.$$

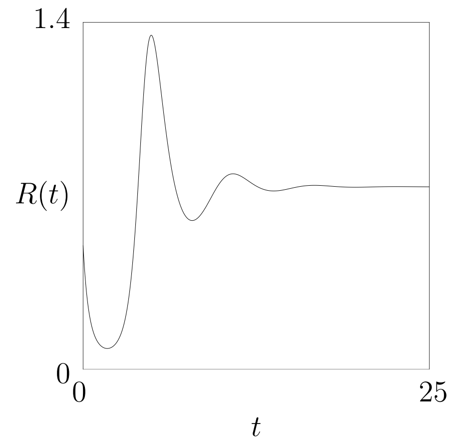

Fig. 2: Level of the Natural Resource over time of the three 5 node leaderless star graphs shown in Figure 1.

The time history of the resource level was identical for all 3 systems and is shown in Figure 2. This is due to the fact that the state of the resource is based on aggregate consumption through the $R(\tau)\sum_{i=1}^{\mathrm{n}} e_i(\tau)$ term in Equation 1. The leaderless condition causes the social component to be canceled out when considered in aggregate. Therefore the resource usage is dependent only on the attributions and scarcity thresholds which are identical for the 3 systems. This can also be seen from Lemma 2 which shows that the equilibrium value of the resource is $\dfrac{\sum_{i=1}^{n} \mathrm{a}_i \mathrm{R}_i}{\sum_{j=1}^{n} \mathrm{a}_j} = 0.74$.

While the resource behavior is identical, the individual usage across networks can be quite different. The individual usages for each of the 3 systems are shown in Figure 3. In Figure 3a and 3b the levels of consumption effort remains low, with those agents that have high scarcity thresholds and environmental attribution contributing negative effort to balance the usage of the other agents. In Figure 3c, the presence of non-cooperative agents drives the scale of individual resource effort to be an order of magnitude larger.

V. Discussion and Conclusion

This paper has shown a number of interesting properties of leaderless networks: that they can exist on arbitrary network topologies, that they are stable under mild bounds on the agent's scarcity thresholds and that on an individual level the consumption effort can be quite different even with the leaderless assumption. Together these results have broadened the understanding of leaderless consumption networks, laying the foundation for the application of this model in large scale networks and the possibility of using this model to design intervention strategies at a community level.

There are other aspects of the behavior of this model worth mentioning. It is interesting to note that the form of the individual consumption effort in Eq. (2) is similar to the consensus dynamic [13], [14], however in leaderless consumption networks the individual consumptions do not

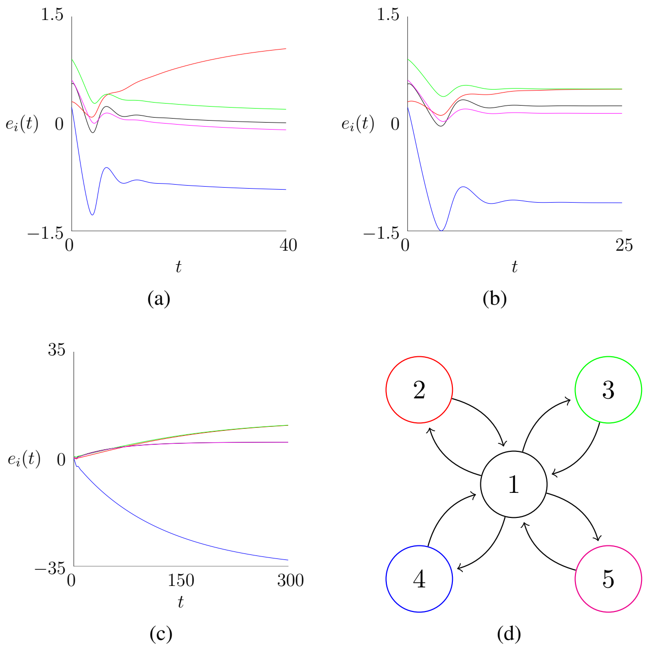

Fig. 3: Evolution of Individual Resource Consumption for 3 Leader-Less Star Graphs: (3a) the random weighted graph shown in (1a). (3b) the uniform weighted graph shown in (1b). (3c) the skew weighted graph shown in (1c). (3d) maps the position of the nodes to trajectories

converge to a single uniform steady-state usage, as would occur under consensus dynamics.

To see why this behavior occurs, recall that the model allows negative resource effort. Here, the stability of the equilibrium point requires that an agent (here agent 4 as shown in Figure 3) contributes resource to ensure balance with the usage of the other agents. As the network is leaderless, the equilibrium behavior of the system depends on the scarcity thresholds. Agent 4 has a scarcity threshold, $R_4 = 1.17$, which is significantly higher than the thresholds of its neighbors. This higher threshold drives the agent to balance out the usage of the agents that have lower thresholds and which therefore consume the resource.

While this system is stable, as shown by Theorem 1 and displayed in Figure 2, this system level behavior would be worrying as the progress of a natural resource. Imagine, for example, the panic of a populace if the level of the local water reservoir were to change as indicated in Figure 2: The reservoir shifts quickly from being almost empty to overflowing and then starts heading back down towards empty before reaching equilibrium.

This points to the fact that stability, while vital for understanding the behavior of a system, is not the only property of a natural resource system which must be understood. There are other questions, those related to sustainability, which must be addressed about these models before they are used to inform decision making in resource governance problems. For example, can humans use this resource in the short term without risking the depletion of the resource in the long term? Future work is required to bridge this gap between stability tools and the characterization of an ecological system as sustainable.

References

[1] Fikret Berkes, Johan Colding, and Carl Folke. *Navigating social-ecological systems: building resilience for complexity and change*. Cambridge University Press, 2008.

[2] Jianguo Liu, Thomas Dietz, Stephen R. Carpenter, Marina Alberti, Carl Folke, Emilio Moran, Alice N. Pell, Peter Deadman, Timothy Kratz, Jane Lubchenco, Elinor Ostrom, Zhiyun Ouyang, William Provencher, Charles L. Redman, Stephen H. Schneider, and William W. Taylor. Complexity of coupled human and natural systems. *Science*, 2007.

[3] Robert M Solow. *Growth theory : an exposition*. Oxford University Press, 2nd ed.. edition, 2000.

[4] Talha Manzoor, Elena Rovenskaya, and Abubakr Muhammad. Game-theoretic insights into the role of environmentalism and social-ecological relevance: A cognitive model of resource consumption. *Ecological Modelling*, 340:74–85, 2016.

[5] Talha Manzoor, Elena Rovenskaya, and Abubakr Muhammad. Structural effects and aggregation in a social-network model of natural resource consumption. *Proc. of the 20th IFAC World Congress*, 2017.

[6] Talha Manzoor, Elena Rovenskaya, Alexey Davydov, and Abubakr Muhammad. Learning through fictitious play in a game-theoretic model of natural resource consumption. *IEEE Control Systems Letters*, 2(1):163–168, 2018.

[7] H Scott Gordon. The economic theory of a common-property resource: the fishery. In *Classic Papers in Natural Resource Economics*, pages 178–203. Springer, 1954.

[8] Roger Perman. *Natural resource and environmental economics*. Pearson Education, 2003.

[9] Sze-Bi Hsu. On global stability of a predator-prey system. *Mathematical Biosciences*, 39(1-2):1–10, 1978.

[10] HI Freedman and JW-H So. Global stability and persistence of simple food chains. *Mathematical biosciences*, 76(1):69–86, 1985.

[11] Chuang-Hsiung Chiu. Lyapunov functions for the global stability of competing predators. *Journal of mathematical analysis and applications*, 230(1):232–241, 1999.

[12] Sze-Bi Hsu. A survey of constructing lyapunov functions for mathematical models in population biology. *Taiwanese Journal of Mathematics*, pages 151–173, 2005.

[13] Robert P Abelson. Mathematical models of the distribution of attitudes under controversy. *Contributions to Mathematical Psychology*, 1964.

[14] Mehran Mesbahi and Magnus Egerstedt. *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.

[15] Anahita Mirtabatabaei, Peng Jia, and Francesco Bullo. Eulerian opinion dynamics with bounded confidence and exogenous inputs. *SIAM Journal on Applied Dynamical Systems*, 13(1):425–446, 2014.

[16] Giovanni Russo and Jean Jacques E Slotine. Global convergence of quorum-sensing networks. *Physical Review E*, 82(4):041919, 2010.

[17] Elinor Ostrom. *Governing the commons*. Cambridge university press, 2015.

[18] Julio Videras. Social networks and the environment. *Annu. Rev. Resour. Econ.*, 5(1):211–226, 2013.

[19] Christina Prell and Örjan Bodin. *Social Networks and Natural Resource Management: Uncovering the social fabric of environmental governance*. Cambridge University Press, 2011.

[20] Leon Festinger. A theory of social comparison processes. *Human relations*, 7(2):117–140, 1954.

[21] Hans-Joachim Mosler and Wernher M Brucks. Integrating commons dilemma findings in a general dynamic model of cooperative behavior in resource crises. *European Journal of Social Psychology*, 33(1):119–133, 2003.

[22] Hassan Khalil. *Nonlinear Systems*. Prentice Hall, 2002.

[23] Roger A Horn and Charles R Johnson. *Matrix analysis*. Cambridge University Press, 1990.

[24] Réka Albert and Albert-László Barabási. Statistical mechanics of complex networks. *Reviews of Modern Physics*, 74(1):47, 2002.

[25] Duncan J Watts and Steven H Strogatz. Collective dynamics of small-world networks. *Nature*, 393(6684):440, 1998.

[26] Mark Newman. *Networks: An Introduction*. Oxford University Press, 2010.