Design of Sustainable Resource Consumption Networks

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Abstract— **This paper considers the design of networks for sustainability in the context of a socio-ecological model of natural resource consumption. Recent work has developed a notion of sustainability motivated by the ecological modelling literature and a set of conditions on the network structure & system parameters that ensure that this sustainability definition is satisfied. This paper translates these sustainability criteria into an optimization problem that optimizes both a network's topology and its interaction weights to make the sustainability time horizon as long as possible. This problem treats system stability as a constraint, and it enforces an "edge budget" for the network, which reflects realistic resource limitations by limiting the number of edges that a network can contain. The introduced optimization problem is then solved analytically for a network of homogeneous agents, and numerical results are shown for heterogeneous agents. Finally the derived optimal network topologies are shown to have varying impacts on the behavior of the resource consumption model. Together, these results suggest that homogeneity in the underlying network structure promotes sustainability in the sense defined herein.**

I. INTRODUCTION

Many complex systems such as river ecologies, fisheries, irrigation networks, forests and wetlands are characterized by the interactions between the biophysical environment and human societies. Human activity impacts natural ecologies via pollution, climate change, resource depletion, renewal and engineered modifications. Conversely, the scarcity or excess of a natural resource shapes human behavior in complex ways. These interactions, often in the form of feedback loops between environment and society are studied under frameworks of socio-ecological systems [**?**], coupled human-natural systems [**?**], and various others [**?**], [**?**].

We study here the sustainability of common pool resources, in which many agents have access to a shared resource. A key phenomenon in such society coupled resources is the so-called tragedy of the commons [**?**], whereby human agents with unrestricted access to a common pool resource find no economic incentive for conservation, leading to rapid exhaustion of the resource from collective overuse. The tragedy lies in the fact, that despite full knowledge of this eventuality, agents find themselves compelled to behave myopically. This is one instance of the more general problem of social dilemmas [**?**]. Game theoretic treatment of such phenomena is demonstrated in the equilibria of some canonical non-cooperative games [**?**] and in related models in evolutionary game-theory [**?**], [**?**]. A political science perspective is given by Olson's Logic of Collective Action [**?**] for similar situations where individual self-interest leads to collectively sub-optimal outcomes in large groups.

The tragedy narrative discussed above has been seriously challenged by groundbreaking work on socio-ecological systems, led by the 2009 Nobel economics prize winner, Professor Elinor Ostrom [**?**]. The central point of her work is that tragedies occur only in situations where agents remain isolated from each other, which rarely occurs in the real world [**?**]. This work suggests that social connections across both individuals and institutions play a key role in the formation of sustainable communities [**?**]. The importance of the underlying social network has also been highlighted in other experimental studies (for instance, [**?**], [**?**], [**?**]).

While this body of work provides clear evidence for the importance of societal connectivity for promoting sustainability, it does not prescribe how beneficial outcomes depend on network structure. This is the research gap that we begin to address in this paper by borrowing tools from networked control systems, distributed optimization and cyber-physical-social systems. Specifically, we attempt to discover network structures that avert tragedy and promote sustainability in a particular analytic model of socio-ecological systems [**?**] with the hope that the lessons learned can be applied to other coupled human-environment systems.

Understanding the interplay between network structure and dynamics that evolve over that structure has been studied extensively in the controls and network science literatures [**?**], [**?**], see for instance the work relating network structure to controllability properties [**?**], [**?**]. Often a network structure is assumed which then drives behavior, with significantly less attention devoted to designing network structures to achieve certain properties. One notable line of "network design" work pertains to designing networks of oscillators to ensure synchronization behavior [**?**], [**?**], [**?**], including in sensor network design [**?**].

In this paper, we consider network designs that promote sustainability. While the network is primarily formed through social connections between consumers, these connections are often supported through physical entities, e.g. physical connections such as canals for water resources, which are subject to resource limitations. We account for these restrictions by enforcing an "edge budget" that upper-bounds the number of edges a network can have. The constraint appears as a 0-norm bound on a matrix of edge weights. Problems with 0-

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norm terms have been considered in the compressed sensing literature, e.g., [?], [?], [?]. It is common to replace a non-convex 0-norm term with a convex 1-norm regularization term, which has been shown to give sparse solutions in many cases [?]. However, in a physical network even an edge with very small weights can be costly, and thus we retain the 0-norm representation. Despite its non-convexity, we will see that surprisingly good numerical results can be attained here.

Recent work by the authors [?] introduced a formal definition of sustainability based on the ecological modelling literature as well as conditions for stability and sustainability of resource consumption. This work was descriptive, asking if a given realization of the socio-ecological model was sustainable. Here we extend this characterization to consider a design question: for a given population of agents, what is the most sustainable network structure?

The rest of the paper is organized as follows. Section II describes the considered socio-ecological model and Section III reviews and extends results on the sustainability of the model. Section IV presents the optimization problem, with analytical results in Section V. Section VI shows numerical results and the behavior of the socio-ecological model. The paper concludes in Section VII.

II. Network Consumption Model

Here we define the dynamics of the coupled socio-ecological system. The model consists of two parts: the ecological sub-model, describing the dynamics of the resource, and the social sub-model, describing the consumption dynamics of individual agents. We present a non-dimensionalized version of the model that is subsequently used in this study. The interested reader is referred to the initial presentation [?] and subsequent analyses [?], [?], [?] for full details on the foundations of the model.

The model considers a single natural resource which is consumed by a population of n individuals. The resource dynamics are governed by the standard model of logistic growth. The dynamics are given by $\dot{x} = (1-x)x - x\sum_{i=1}^{n} y_i$, where x is the quantity of resource normalize with respect to its carrying capacity and y_i is the consumption of agent i normalized with respect to the growth rate of the resource.

Results from the ecological modelling literature suggest that consumers balance social and ecological information in making consumption decisions, for an extended discussion and motivation of the model see [?]. The impact of these two factors on agent consumption is modeled as

$$\dot{y}_i = b_i \left(\alpha_i (x - \rho_i) + \nu_i \sum_{j=1}^{n} \omega_{ij}(y_j - y_i) \right),$$

where $i \in \{1, \ldots, n\}$. Here $\alpha_i \in (0,1)$ and $\nu_i \in (0,1)$ are the weights representing the relative importance that i gives to the ecological and social factors respectively. Due to the normalization it holds that $\alpha_i + \nu_i = 1$. $\rho_i \in \mathbb{R}$ is called the environmentalism of i and represents the threshold below which i considers the resource to be scarce and above which she considers it to be abundant. $b_i > 0$ is the sensitivity of i and denotes i's openness to change in her consumption. Finally, $\omega_{ij} \in [0,1]$ is the strength of the social tie directed from j to i, and we assume the following:

Assumption 1: For all $i \in \{1, \ldots, n\}$, we have $\omega_{ii} = 0$ and $\sum_{j=1}^{n} \omega_{ij} = 1$. △

A. Coupled Socio-ecological System

Here we give the complete system describing the coupled dynamics of the resource growth and consumer behavior in compact notation. Define the vector $y = (y_1, y_2, \ldots, y_n)^\top$ as the vector containing all individual consumptions, and the vector $\rho = \text{diag}(\rho_1, \ldots, \rho_n)$ as the vector containing all scarcity thresholds. Further define the matrices $A = \text{diag}(\alpha_1, \ldots, \alpha_n)$, $B = \text{diag}(b_1, \ldots, b_n)$, $V = \text{diag}(\nu_1, \ldots, \nu_n)$, and

$$T = \begin{pmatrix} 1 & -\omega_{12} & -\omega_{13} & \cdots & -\omega_{1n} \\ -\omega_{21} & 1 & -\omega_{23} & \cdots & -\omega_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\omega_{n1} & -\omega_{n2} & -\omega_{n3} & \cdots & 1 \end{pmatrix}.$$

The coupled socio-ecological system is then of the form

$$\dot{x} = (1-x)x - x\mathbf{1}^\top y,$$
$$\dot{y} = B\,A(x\mathbf{1} - \rho) - B\,V\,T\,y.$$

Finally, we define the parameter $\theta_i = \frac{\nu_i}{\alpha_i}$ which can be regarded as a measure of how social i is. The matrix $\Theta = \text{diag}(\theta_1, \theta_2, \ldots, \theta_n)$.

III. Sustainability Bounds

In this section we review some key results from [?] which motivate the problem discussed in the paper. In [?], stability was shown under the following three assumptions:

Assumption 2: The matrix $\left(A\mathbf{1}\mathbf{1}^\top + VT\right)^{-1}$ exists and $\mathbf{1}^\top \left(A\mathbf{1}\mathbf{1}^\top + VT\right)^{-1} A(\mathbf{1} - \rho) < 1$. △

Assumption 3: For all $i \in \{1, \ldots, n\}$, $\theta_i \geq \sum_{k=1, k\neq i}^{n} \omega_{ki}\theta_k$. △

This condition is sufficient to imply $T^\top \Theta + \Theta T \succeq 0$ (via Gershgorin's Circle Theorem), which was used to show stability in [?]. Assumption 3 has the advantage of being locally checkable by agent i. Conversely, $T^\top \Theta + \Theta T \succeq 0$ enforces the stability criteria at the network-level. It will be used in formulating network-level design problems below.

Assumption 4: The graph connecting agents is strongly connected. △

Also in [?], the stability results were extended to consider a notion of sustainability in which the state of the resource was guaranteed to satisfy $v_{min} \leq x \leq v_{max}$ and $d_{min} \leq \dot{x} \leq d_{max}$ for $t \in [0, t_{max}]$. In this paper, the parameters $v_{min}, v_{max}, d_{min}, d_{max}$ are considered intrinsic to the resource and we seek to understand how the underlying graph topology relates to the sustainability time horizon t_{max}. To do so requires a novel lemma:

Lemma 1: Consider $\gamma_0(T) = \ln\left(1 - \mathbf{1}^\top \left(A\mathbf{1}\mathbf{1}^\top + VT\right)^{-1} A(\mathbf{1} - \rho)\right)$. Under Assumption 1 and 2, there exist constants a and b such that $\gamma_0(T) \in [a, b], \ \forall T$.

Proof: Under Assumption 1, the individual elements of T are in $[0,1]$, and the set of admissible matrices T is therefore a compact subset in $\mathbb{R}^{n\times n}$. Under Assumption 2,

$\gamma_0(\cdot) : \mathbb{R}^{n\times n} \to \mathbb{R}$ is a continuous function of T, which implies that it maps the compact set of admissible matrices T to a compact set in R, proving the lemma. ■

Note also that from the structure of γ_0, we expect variations in T to have a relatively small impact on the resulting γ_0. This further justifies recasting the sustainability bounds of [**?**] in terms of the result of Lemma 1.

Assumption 5: It holds that $v_{min} < v_{max}$, $v_{max} > 0$, $v_{min} < v(0) < v_{max}$, and $d_{min} < 0 < d_{max}$. △

Theorem 1: (from [**?**]) Let Assumption 5 hold. Define $\xi_1 = e^a(e^{v_{max}} - 1)$, $\xi_2 = v(0) - t_{max}e^b(e^{v_{max}} - 1) - v_{min}$, $\xi_3 = d_{max} + e^a(e^{v_{min}} - 1)$, and $\xi_4 = -d_{min} - e^b(e^{v_{max}} - 1)$. We assume that, for all $i \in \{1,2,3,4\}$, $\xi_i > C_1$, where $C_1 = \|w(0)\|_1 + t_{max}e^{\gamma_0}(e^{v_{max}} - 1)\sum_{i=1}^n b_i\alpha_i$. Then the system is sustainable if

$$\|T\|_1 \le \frac{1}{\beta t_{max}} \log\left(\frac{\min\limits_{i\in\{1,2,3,4\}} \xi_i}{C_1}\right), \tag{1}$$

where $\beta = \max_i b_i\nu_i$.

Now the main theorem of the paper can be proven.

Theorem 2: For a given set of $\{v_{max}, v_{min}, d_{max}, d_{min}\}$ and model parameters, the sustainability bound $t_{\max}$ in Eq. (1) can be maximized if and only if $\|T\|_1$ is minimized.

Proof: The network structure enters the sustainability bound of Eq. (1) solely through the term $\|T\|_1$. To see that $\|T\|_1$ should be minimized, consider the function

$$g(t_{max}) = \frac{1}{\beta t_{max}} \log\left(\frac{\min\limits_{i\in\{1,2,3,4\}} \xi_i}{C_1}\right).$$

The function $g(t_{max})$ is a positive, decreasing function of t_{max} when $\xi_i > C_1$. The function $g(t_{max})$ takes two basic forms based on which of the ξ_i is the minimum. If $\xi_1, \xi_3,$ or ξ_4 is the minimum then $g(t_{max}) = \frac{1}{\beta t_{max}}\log\left(\frac{c_1}{c_2 + c_3 t_{max}}\right)$, where $c_i, i \in \{1,2,3\}$, are positive constants, with $c_1 = \xi_i,\ i \in \{1,3,4\}$ depending on the minimum, $c_2 = \|w(0)\|_1$, and $c_3 = e^{\gamma_0}(e^{v_{max}} - 1)\sum_{i=1}^n b_i\alpha_i$. This is manifestly a positive, decreasing function of t_{max}. If ξ_2 is the minimum, then $g(t_{max}) = \frac{1}{\beta t_{max}}\log\left(\frac{c_4 - c_5 t_{max}}{c_2 + c_3 t_{max}}\right)$, where $c_i, i \in \{2,3,4,5\}$ are positive constants with $c_4 = v(0) - v_{\min}$ and $c_5 = e^{\gamma_0}(e^{v_{max}} - 1)$. The term $\frac{(c_4 - c_5 t_{max})}{c_2 + c_3 t_{max}}$ is positive when $\xi_2 > C_1$ and has derivative $\frac{-(c_2c_5 + c_4c_5)}{(c_2 + c_3 t_{max})^2}$ which is negative and well defined for positive t_{max}, making $g(t_{max})$ a positive decreasing function of t_{max} when $\xi_i > C_1,\ \forall i \in \{1,2,3,4\}$. Returning to Equation (1), as $g(t_{max})$ is positive and decreasing $\|T\|_1$ should be minimized to allow for larger values of t_{max}. ■

We will consider the minimization of $\|T\|_1$ as our primary problem of the paper.

IV. Network Optimization

This section formalizes the network optimization problem which will be solved analytically and numerically in the remainder of the paper. Below, we use J to denote the $n \times n$ matrix of all ones. We begin with the following lemma.

Lemma 2: Let $T \in \mathbb{R}^{n\times n}$ denote the weights matrix as defined in Section II. Then the matrix $\frac{1}{2}(T^\top + T)$ is positive semidefinite, has rank $n - 1$, and has nullspace spanned by the vector $\mathbf{1} := (1, 1, \ldots, 1)^\top$.

Proof: See Lemmas 8, 9, and 10 in [**?**]. ■

The following convex constraint gives a necessary condition for strong connectivity. Below, we discuss why we expect this necessary condition to provide useful results.

Theorem 3: The graph $\mathcal{G}$ is strongly connected only if $\frac{1}{2}\left(T + T^\top\right) + J \succ 0$.

Proof: For the sake of contradiction, we assume that there is some $x \neq 0$ such that

$$x^\top\left(\frac{1}{2}(T + T^\top) + J\right)x = 0. \tag{2}$$

Lemma 2 implies that $x^\top(\frac{1}{2}(T + T^\top))x \ge 0$ and it is easy to see that $x^\top Jx \ge 0$. Satisfying Equation (IV) requires $x \neq 0$ to satisfy both $x^\top(\frac{1}{2}(T + T^\top))x = 0$ and $x^\top Jx = 0$ simultaneously. For $x^\top(\frac{1}{2}(T + T^\top))x = 0$, Lemma 2 shows that we must have $x \in \text{span}\{\mathbf{1}\}$, while we have $x^\top Jx = 0$ if and only if $\sum_{i=1}^n x_i = 0$, which we restate as $x^\top\mathbf{1} = 0$. Then satisfying Equation (IV) requires $x \neq 0$ to satisfy both $x \in \text{span}\{\mathbf{1}\}$ and $x \in \text{span}\{\mathbf{1}\}^\perp$, a contradiction. ■

This necessary condition is "almost" sufficient in a precise sense. Using the results of [**?**, Theorem 4.3], it can be shown that a T satisfying the above has at most one isolated strongly connected component; here, "isolated" means that there are no directed edges pointing into the strongly connected component in question. The inequality in Theorem 3 is sufficient to imply weak connectivity, and, it will at least give a weakly connected graph with not more than one isolated strongly connected component. It has the benefit of being convex, and we find that a solver typically generates strongly connected graphs when this constraint is used.

The final constraint which will be enforced is an edge budget. This edge budget reflects many of the realities of network design and as will be seen allows the differentiation between multiple solutions. Enforcing the edge budget can be done by counting non-zero off-diagonal entries of the matrix T, which is the number of non-zero entries of $T - I$. To do so, we adopt the so-called "0-norm", often used is machine learning and compressed sensing applications [**?**], [**?**], [**?**] to promote sparsity in solutions to various problems.

Definition 1: The matrix 0-norm, denoted $\|\cdot\|_0 : \mathbb{R}^{n\times n} \to \mathbb{N}$, is defined as $\|M\|_0 = \left|\{(i,j) \in \{1,\ldots,n\}^2 \mid M_{ij} \neq 0\}\right|$, which is equal to the number of non-zero entries of its argument. ⋄

The 0-norm is not truly a norm (nor is it convex), though it is well-studied and there exist computational tools that accommodate it. The edge budget constraint takes the simple form $\|T - I\|_0 \le B$. Then the overall problem statement is:

Problem 1: Design a resource consumption network with weight matrix T that satisfies the following:

$$\begin{aligned} \text{minimize} \quad & \|T\|_1 \\ \text{subject to} \quad & \frac{1}{2}(T^\top + T) + J \succeq 0 \\ & T^\top\Theta + \Theta T \succeq 0 \\ & \|I - T\|_0 \le B. \end{aligned}$$

$\diamondsuit$

Problem 1 is non-convex due to the non-convex 0-norm constraint. Nonetheless, we will see that analytical and numerical results can produce useful solutions to this problem.

V. Homogeneous Networks

In this section we consider networks that are homogeneous in theta, i.e. $\theta_i = \theta$. It is possible to analytically characterize solutions to Problem 1 for such networks. We begin with some definitions. A graph is *weight balanced* if $\sum_k \omega_{ki} = \sum_j \omega_{ij},\ \forall i$. By Assumption 1, $\sum_j \omega_{ij} = 1,\ \forall i$ so a weight balanced graph will have $\sum_k \omega_{ki} = 1,\ \forall i$. A graph is k-*regular* if each node has in degree k and out degree k. We refer to the *star graph* to describe an n node graph where a single node has in and out degree $n-1$ while all other nodes have in and out degree 1.

Consider the following characterization of $\|T\|_1$.

Lemma 3: Under Assumption 1, it holds that $2 \leq \|T\|_1 \leq n$. Further, a weight balanced graph achieves the minimum and a star graph achieves the maximum.

Proof: To show that $2 \leq \|T\|_1$, note that if the graph is weight balanced then the absolute value of every column sums to 2 which implies $\|T\|_1 = 2$. To see that there can be no smaller value of $\|T\|_1$, suppose there exists a set of weights such that $\|T\|_1 < 2$. Then by definition of the 1-norm and of T, this gives rise to a contradiction as there $\exists i,$ s.t. $\sum_{k\neq i} \omega_{ki} < 1$ which implies $n = \sum_i \sum_j \omega_{ij} < n$.

We now show that $\|T\|_1 \leq n$. Under the assumption that $\sum_j \omega_{ij} = 1, \omega_{ij} \geq 0$, every off diagonal element satisfies $T_{ij} \leq 1, \forall i \neq j$. As $T_{ii} = 1$ and $T \in \mathbb{R}^{n\times n}$, the condition follows directly. In the case of the star graph, there will be a column which is all ones, so $\|T\|_1 = n$. ■

To characterize the optimal solutions to Problem 1, we will show that a class of graphs achieves the minimum possible value of the objective function, that $\|T\|_1 = 2$.

Lemma 4: If $\theta_i = \theta$ for all i, Assumption 3 is satisfied if and only if the graph is weight-balanced.

Proof: (Sufficiency) In Assumption 3, setting $\theta_i = \theta$ for all i gives $1 \geq \sum_{\substack{k=1\\k\neq i}}^{n} \omega_{ki}$. If the graph is weight balanced, then $\sum_{\substack{k=1\\k\neq i}}^{n} \omega_{ki} = 1$ for all i and Assumption 3 is satisfied.

(Necessity) Assume the graph is not weight balanced and Assumption 3 holds. Then there must exist a node i such that $1 > \sum_{\substack{k=1\\k\neq i}}^{n} \omega_{ki}$. Summing over all nodes gives rise to a contradiction as $n > \sum_i \sum_{\substack{k=1\\k\neq i}}^{n} \omega_{ki} = \sum_i \sum_j \omega_{ij} = n$. ■

From Lemma 4, we can see that any connected weight balanced graph will satisfy two of the three constraints of Problem 1. In fact, the structure of T shows that any weight balanced graph will minimize the objective $\|T\|_1$.

Theorem 4: If $\theta_i = \theta$ for all i, then any weight-balanced connected k-regular directed graph with $k \leq \frac{B}{n}$ is an optimal solution to Problem 1.

Proof: Note that a k-regular graph has nk edges, so if $k \leq \frac{B}{n}$ the graph satisfies the edge budget constraint. The graph is weight balanced and connected so Assumption 3 is satisfied by Lemma 4. Assumption 3 is sufficient to show that $T^\top\Theta + \Theta T \succeq 0$. Therefore the described k-regular graph is an optimal solution to Problem 1. ■

These results can be related back to commonly considered graph structures, to show how these structures relate to sustainability in the case of homogeneous agents. Cycle graphs, which are 1- and 2-regular for a directed and bidirectional cycle respectively, easily satisfy Assumption 3 and are optimal solutions to Problem 1 with small numbers of edges. In fact, the 1-regular directed graph is the smallest possible solution with n edges.

Moving away from regular graphs, the line graph can not be an optimal solution as ensuring that the peripheral nodes are weight balanced would disconnect the graph. Similarly, star graphs are particularly poor for sustainability with respect to the objective of Problem 1 as they maximize $\|T\|_1$, as discussed in Lemma 3. This seems to suggest that in the case of homogeneous agents, network structures that allow social influence to be spread between many agents, by being homogeneous in degree distribution, improve sustainability. While the characterization of solutions to Problem 1 is possible in the case of a network with homogeneous agents, heterogeneity in θ can greatly restrict the space of feasible solutions as shown in the following result.

Theorem 5: There exists θ_i such that there is no feasible graph structure under Assumption 3.

Proof: As an example consider the feasible solutions on 3 nodes in the case where $\theta_1 \neq \theta_2 = \theta_3 = \theta$. First consider $\theta_1 = 2\theta$. Assumption 3 leads to three conditions $\theta_1 \geq (\omega_{21} + \omega_{31})\theta$, $\theta \geq \frac{\omega_{12}}{\omega_{21}}\theta_1$, and $\theta \geq \frac{\omega_{13}}{\omega_{31}}\theta_1 = \frac{1-\omega_{12}}{\omega_{31}}\theta_1$.

The first equation is trivially satisfied as $\omega_{21}, \omega_{31} \leq 1$. The second two equations constrain ω_{12} as $\frac{\omega_{21}}{2} \geq \omega_{12} \geq \frac{2-\omega_{31}}{2}$. The only feasible solution is $\omega_{21} = \omega_{31} = 1$ and $\omega_{13} = \omega_{12} = \frac{1}{2}$, i.e. a star graph is the only admissible solution. If instead $\theta_1 = k\theta,\ k > 2$ then there is no graph which satisfies Assumption 3 as the new constraint $\frac{\omega_{21}}{k} \geq \omega_{12} \geq \frac{k-\omega_{31}}{k}$ can not be satisfied with $\omega_{21}, \omega_{31}, \omega_{12} \leq 1$. ■

The simulations shown in the next section show cases where the values of θ_i are similar enough that solutions exist An interesting direction for future work would involve characterizing when these conditions hold, i.e. when agents are so dissimilar that sustainability is not feasible.

VI. Simulations

In this section, we consider two sets of simulations. First the optimization problem 1 is solved numerically for heterogeneous agents. Then the results for homogeneous agents are discussed in the context of the behavior of the socio-ecological system which evolves over the network.

A. Heterogenous Networks

Problem 1 was solved numerically using the YALMIP software package [**?**]. Due to the non-convexity of the problem, the case of a 10 node network is considered here. For a given number of nodes, the matrix Θ and the edge budget B must be supplied to fully constrain the problem.

In the following the values of Θ were randomly selected as $\Theta = \mathrm{diag}([2, 155\ 5, 036\ 5, 306\ 5, 809\ 6, 878\ 11, 681\ 12, 342$

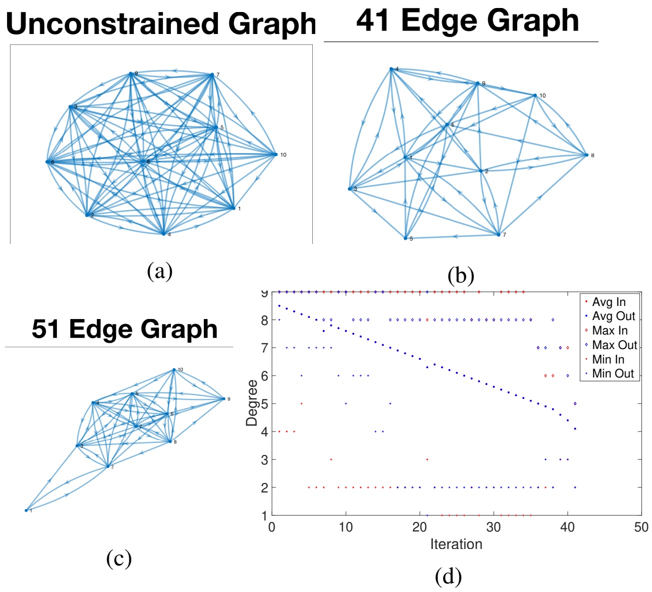

Fig. 1: Graph returned with no edge budget (e.b.) 1a, the most stringent e.b. 1b, and an intermediate e.b. 1c. 1d: Degree change over iterations of the algorithm. The average, max, and min of both in and out degree are shown.

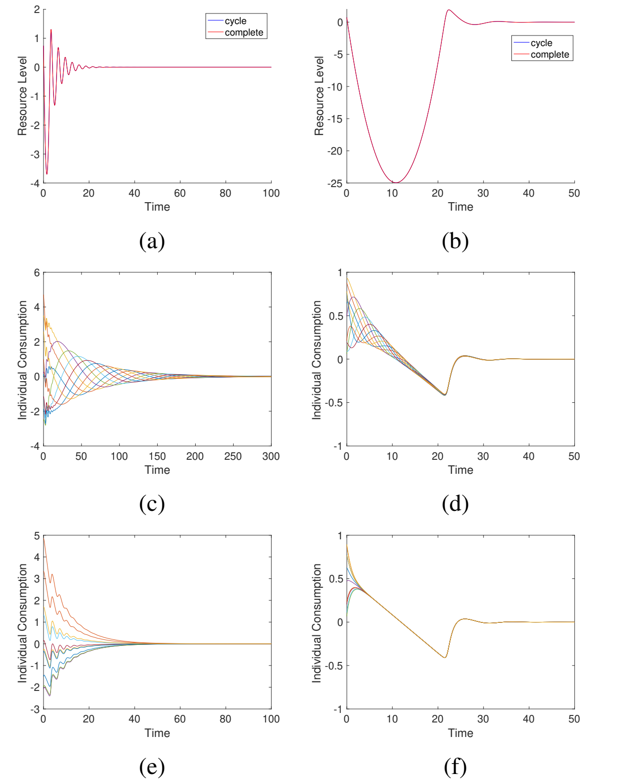

Fig. 2: Resource x for Pro-Ecological 2a and Pro-Social Communities 2b. Individual consumption for Pro-Ecological cycle 2c and complete graphs 2e. Individual consumption for Pro-Social cycle 2d and complete graphs 2f.

$14,605\ 16,488\ 19,653]$). Extensive simulation suggests that larger values of Θ prevented numerical issues with the solver.

To understand how the edge budget affects the solution to Problem 1, the optimization is run multiple times. First it is run without the edge budget to get a baseline number of edges for a given Θ, the result of which is shown in Figure 1a. Once the maximum allowable number of edges is determined from the unconstrained optimization, the optimization is rerun multiple times. On each iteration the edge budget is decreased by one until the solver returns that the problem is no longer feasible. Figure 1 shows example graphs from a representative run and Figure 1d shows how the degree changes over iterations. Each graph was checked for strong connectivity, which was satisfied for all graphs. Figure 1b shows the graph with the smallest number of edges returned with 41 edges. Figure 1c shows an intermediate graph in which a single node (node 1) becomes isolated.

Figure 1d shows that the solver was able to systematically reduce the average degree of the network. At the same time, the maximum and minimum degrees show that this occurred by having nodes with a very small degree. In the limit as the problem became infeasible, the last high degree nodes were removed moving towards a more homogeneous network structure. Taken together with the analytical results, the simulations suggest that for a given edge budget, it is better to provide as homogeneous a network as possible, facilitating as much communication as possible.

B. Resource Behavior

In this section, we return to the dynamics which evolve over the graphs which were determined by the solution of Problem 1. To begin we consider the homogeneous network case, specifically a 10 node graph where $\theta_i = \theta, \forall i \in \{1, 2, \dots, 10\}$. As discussed previously and in [**?**], a given set of thetas can describe a variety of different preferences towards environmental or social information. Here we explore the differences in behavior in a pro-social society ($\theta = .1$) and a pro-ecological society ($\theta = 10$). The underlying graph was taken to be either a directed cycle graph (1-regular graph) or a complete graph (9-regular graph), which are both optimal solutions to Problem 1 by Theorem 1. The remaining parameters were taken to be $b_i = 1, \forall i$, randomly selected starting conditions and thresholds $\rho = [0.244\ 0.929\ 0.350\ 0.197\ 0.251\ 0.616\ 0.473\ 0.352\ 0.831\ 0.585]$.

Figure 2 shows the resource level behavior for both the Pro-Ecological and Pro-Social communities. The underlying graph structure does not affect the resource consumption in both the Pro-Ecological (2a) and Pro-Social case (2b). The Pro-Social society has a much slower response time, as information about the consumption of network neighbors is prioritized. At the individual consumption level however the underlying graph structure has a large impact. For both the Pro-Social and Pro-Ecological societies the directed cycle graph produces significant fluctuations as information about consumption behavior takes much longer to flow through the network. These simulations suggest that the full edge budget should be used to facilitate communication. Future work is required to characterize an agent based notion of sustainability which fully captures the differences between pro-social and pro-ecological behavior on these graphs.

VII. Conclusion

This paper considered designing the underlying social network of a population to ensure sustainability The problem was formalized and analytical results were shown in the case of a population that is homogeneous w.r.t the preference of the agents to social information relative to ecological information. Numerical results showed that even in heterogeneous populations, homogeneity in the social network was beneficial for sustainability. In the literature, opinions on the effect of heterogeneity on successful resource management are highly variable [**?**], [**?**]. Indeed past work on the same model [**?**] showed that heterogeneity in the agent's social relevance decreases Tragicness (a notion related to the distance between the tragedy of the commons and the current state of the community). However the results included in this paper suggest that heterogeneity in social relevance decreases the time horizon over which the system is sustainable. While our study does not provide a straight-forward verdict on the role of heterogeneity, it does indicate that heterogeneity needs to be exposed to a more thorough classification to correctly establish its correlation with successful resource management.