

# Stability and Sustainability of a Networked Resource Consumption Model

Matthew T. Hale<sup>\*</sup>, Sebastian F. Ruf<sup>†</sup>, Talha Manzoor<sup>‡</sup>, Abubakr Muhammad<sup>§</sup>



**Abstract**—In this paper, we examine both stability and sustainability of a network-based model of natural resource consumption. Stability is studied from a dynamical systems perspective, though we argue that sustainability is a fundamentally different notion from stability in social-ecological systems. Accordingly, we also present a criterion for sustainability that is guided by the existing literature on sustainable development. Assuming a generic social network of consuming agents' interactions, we derive sufficient conditions for both the stability and sustainability of the model as constraints on the network structure itself. We complement these analytical results with numerical simulations and discuss the implications of our findings for policy-making for sustainable resource governance.

## 1 INTRODUCTION

Sustainability of social-ecological systems has been a subject of considerable interest for some time [1]. However, despite its importance, sustainability of this kind is still far from being adequately defined in a rigorous context. The confusion stems directly from the definition of sustainable development itself, given by the Brundtland Commission of the World Commission on Environment and Development (WCED) [2], which leaves much room for interpretation. This has led to a string of research spanning across multiple disciplines in search of a formal definition of the concept [3], [4], [5].

From a dynamical systems perspective, sustainability has often been linked to developments in [6], [7], and sometimes even used interchangeably with stability in social-ecological systems [8]. We argue that while stability is indeed relevant in this setting, it is a purely dynamical systems property that does not

appropriately account for the ecological nature of the system. Thus a separate formulation of sustainability is needed. This need is underscored by the rising interest of the controls community not only in social-ecological systems, but also in other systems lying at the interface of technological, ecological, and social sciences. Such systems are collectively termed “cyber-physical social systems” [9].

In this paper we study the stability and sustainability of a social-ecological system whose model was first presented and studied in [10]. The model is for a natural resource being harvested by a fixed number of consuming agents. While deciding their consumption, the agents not only take information about the resource into account, but also incorporate information about the consumption of other neighboring agents. The social network of the population thus plays a critical role in determining overall system behavior, and it must also affect sustainability in any sense that is considered. Indeed, it has been found [1] that the ability of a population to sustainably manage its resources at the community level is highly dependent on the underlying social network of that community.

Accordingly, the insights provided by the network structure have resulted in an increase in the application of network analysis tools to natural resource systems [11], [12]. For instance, social structure has been observed to be strongly correlated with pro-environmental behavior in social-ecological settings [13]. Information about the structure of the network has also been exploited in the past to identify elements critical to sustainable governance of a resource (for some representative studies, see [14], [15], [16]). Previously, our selected model of resource consumption has been studied only under restrictive assumptions on the network topology [17].

In this paper, we maintain a generic network structure to fully understand the effects of the social network on sustainability of its resource. A complete characterization of sustainable communities remains an open problem for the scientific community. This has

<sup>\*</sup>Department of Mechanical and Aerospace Engineering, University of Florida, Gainesville, FL, USA. Email: matthewhale@ufl.edu.

<sup>†</sup>Center for Complex Network Research and Department of Psychology, Northeastern University, Boston, MA, USA. Email: sebastianfruf@gmail.com.

<sup>‡</sup>Department of Electrical Engineering, Namal College, Mianwali, Pakistan. Email: talha@namal.edu.pk.

<sup>§</sup>Department of Electrical Engineering, Lahore University of Management Science, Lahore, Pakistan. Email: abubakr@lums.edu.pk.

been attributed in part to the highly inter-disciplinary nature of the field and a lack of integrative studies to unify the research that remains scattered among different strands of work [18]. In this paper we devise a criterion for sustainability that is subsequently applied to our network model of resource consumption. This results in structural conditions on the network topology which, we believe, contributes a step towards uncovering the structural characteristics of sustainable societies.

In what follows, we develop a sustainability criterion that draws from the literature on sustainable development. This criterion is based on a rigorous definition of sustainability that we introduce, and we show how the enforcement of system sustainability is translated into conditions on the network topology and its associated parameters. We apply this criterion to a network model of resource-consuming agents and compare it with sufficient conditions for stability. To that end, we also present a novel proof of stability for the full  $n$ -agent model which was previously known to be stable only for 2 agents [17]. The contributions of this paper thus consist of the stability proof for the general  $n$ -agent model and the sustainability criterion we develop, together with the resulting analyses and implications for resource consumption networks that we discuss.

The remainder of the paper is organized as follows. Section 2 first provides the model of interest. Then, Section 3 presents sufficient conditions for stability of the model and a Lyapunov-based stability proof. Section 4 next provides a commentary on the sustainability literature originating from different disciplines. Following guidelines from this exposition, we also present our mathematical definition of sustainability. In Section 5, we obtain sufficient conditions for system sustainability in the form of constraints on the structure of its underlying social network. Section 6 presents simulations of both stable and sustainable networks along with a discussion on the implications for policy making. We conclude in Section 7.

## 2 BACKGROUND AND NETWORK MODEL

This section describes the system model of interest and its coupled resource and consumption dynamics. The model was first presented in [10] and subsequently studied in [17], [19], [20], [21]. For full details on the environmental and social-psychological foundations of the model, we refer the reader to the sources mentioned above. Here we present a non-dimensionalized version of the model (also introduced in [10]) that not only has a reduced parameter space, but also allows more straightforward interpretations of the results relative to the original model.

The setting is that of a single natural resource whose stock renews according to the standard model

of logistic growth [22]. The resource has an associated carrying capacity and intrinsic growth rate that affect the evolution of its value over time. Let  $x(t) > 0$  represent the resource quantity at time  $t$  relative to its carrying capacity. Thus  $x(t) = 1$  implies that at time  $t$  the resource stock is at its environmental carrying capacity. The resource is harvested by a consuming population consisting of  $n$  agents. Each agent harvests the resource by exerting some effort, and, in this model, this effort constitutes all the different dimensions of harvesting activity. In fishing for instance, it can capture the number of boats deployed, their efficiency, the number of days fishing is undertaken, etc. [22]

Let  $y_i(t) \in \mathbb{R}$  represent the consumption effort of agent  $i$  relative to the rate of growth of the resource; a consumption effort of  $y_i(t) = 1$  implies a consumption rate that is equal to the intrinsic growth rate of the resource. The resource dynamics are then given by

$$\dot{x}(t) = (1 - x(t))x(t) - x(t) \sum_{i=1}^n y_i(t),$$

where  $i \in \{1, \dots, n\}$  indexes the agents in the network. Note that the model permits a negative consumption rate. From the dynamics of the resource it is evident that this may result in  $x(t)$  taking on values greater than unity, which corresponds to the stock level crossing the natural carrying capacity of the environment. In [10] we discuss the physical interpretations of this phenomenon in detail. While a positive consumption rate represents an effort to consume the resource (thereby decreasing its stock), a negative rate corresponds to an action to increase the stock of the resource. This includes actions that directly increase the stock (such as planting trees or breeding fish) as well as more indirect actions (such as restoration of soil fertility or a favorable distribution of gear to local fishermen). These actions may also be directed towards artificially increasing the carrying capacity, e.g., de-silting water canals, increasing the land available for forest growth, shifting to more intensive farming techniques, etc.

Next we present the dynamics of the individual consumption efforts  $y_i(t)$ . According to Festinger's theory of social comparison processes [23], while making decisions, humans incorporate both objective information and social information. In the context of resource consumption [24], the objective information is interpreted as information on the state of the resource, i.e., whether it is abundant or scarce. We model this through the scarcity threshold  $\rho_i > 0$  associated with agent  $i$ . The state of the resource is then captured by the ecological factor  $(x(t) - \rho_i(t))$ . A negative ecological factor indicates that agent  $i$  perceives the resource as scarce, whereas a positive factor indicates that agent  $i$  perceives the resource to be abundant. The

social information is interpreted as the difference between agent  $i$ 's consumption level and the consumption levels of other neighboring agents. This is represented by the social factor  $\sum_{j=1}^n \omega_{ij} (y_j(t) - y_i(t))$ , where  $\omega_{ij} > 0$  is the directed tie-strength between agents  $i$  and  $j$  and represents the influence that agent  $j$  has on agent  $i$ 's consumption. Furthermore  $\omega_{ii} = 0$  and  $\sum_{j=1}^n \omega_{ij} = 1$  for all  $i$ .

The rate of change of consumption effort is then represented as a weighted sum of the ecological and social factors. These are weighed by the ecological and social weights  $\alpha_i \in [0, 1]$  and  $\nu_i \in [0, 1]$ , respectively (these terms are also referred to as the social and ecological relevances of agent  $i$  below). Since the weighing of both factors in the final decision-making is observed to be one-dimensional [23], the weights sum to one, i.e.,  $\alpha_i + \nu_i = 1$  for all  $i$ . Thus the consumption dynamics are given by

$$\dot{y}_i(t) = b_i \left( \alpha_i (x(t) - \rho_i) - \nu_i \sum_{j=1}^n \omega_{ij} (y_i(t) - y_j(t)) \right),$$

where  $b_i > 0$  is called the sensitivity of agent  $i$  and represents the readiness of agent  $i$  to change her consumption.

Based on the above, the coupled dynamics of the resource stock and consumption effort are

$$\begin{aligned} \dot{x}(t) &= (1 - x(t))x(t) - x(t) \sum_{i=1}^n y_i(t), \\ \dot{y}_i(t) &= b_i \left( \alpha_i (x(t) - \rho_i) - \nu_i \sum_{j=1}^n \omega_{ij} (y_i(t) - y_j(t)) \right), \end{aligned} \quad (1)$$

where  $i \in \{1, \dots, n\}$ . In Equation (1),  $t$  is non-dimensional time which is normalized with respect to the resource growth rate. In [10], we have discussed how this normalization reduces the dimension of the parameter space relative to the original model, along with interpretations of the various variables and their domains, and we refer the reader to that reference for an extended discussion of the subject.

### 3 STABILITY OF NETWORKED RESOURCE CONSUMPTION

This section proves that the network consumption model in Section 2 is asymptotically stable. We first rewrite the dynamics as an equivalent aggregate-level resource and consumption system and then transform this system into one whose equilibrium is at the origin. Then we provide a Lyapunov-based stability proof. Below, we use  $\text{diag}(a_1, \dots, a_n)$  to denote the diagonal matrix with the scalars  $a_1$  through  $a_n$  on its main diagonal.

#### 3.1 Aggregate Network Dynamics

We define the states  $\gamma = \ln x$  and  $y = (y_1, \dots, y_n)^T$  along with the matrices  $A = \text{diag}(\alpha_1, \dots, \alpha_n)$ ,  $B = \text{diag}(b_1, \dots, b_n)$ , and  $V = \text{diag}(\nu_1, \dots, \nu_n)$ . We further define the vector  $\rho = (\rho_1, \dots, \rho_n)^T$  and the matrix of weights

$$T = \begin{pmatrix} 1 & -\omega_{12} & -\omega_{13} & \cdots & -\omega_{1n} \\ -\omega_{21} & 1 & -\omega_{23} & \cdots & -\omega_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\omega_{n1} & -\omega_{n2} & -\omega_{n3} & \cdots & 1 \end{pmatrix}.$$

We further define  $\theta_i = \frac{\nu_i}{\alpha_i}$ , which we will use below.

Differentiating  $\gamma$  and  $y$  with respect to time gives

$$\begin{aligned} \dot{\gamma} &= \frac{\dot{x}}{x} = 1 - x - \mathbb{1}^T y \\ \dot{y} &= BA(e^\gamma \mathbb{1} - \rho) - BVTy. \end{aligned}$$

In this form, we impose the following assumptions.

**Assumption 1.** The matrix  $(A\mathbb{1}\mathbb{1}^T + VT)^{-1}$  exists and

$$\mathbb{1}^T (A\mathbb{1}\mathbb{1}^T + VT)^{-1} A(\mathbb{1} - \rho) < 1. \quad \triangle$$

Below, Assumption 1 will be used to insure that  $\gamma_0$  and  $y_0$ , the respective equilibria of the  $\gamma$  and  $y$  dynamics, are well-defined.

**Assumption 2.** The underlying graph connecting agents is strongly connected.  $\triangle$

Assumption 2 is a common assumption in multi-agent networks. Strong connectivity implies, roughly, that there is "enough" interaction among agents to drive the state of the system toward its equilibrium point, and this notion will be made precise in Theorem 1.

**Assumption 3.** For all agent indices  $i \in \{1, \dots, n\}$ ,

$$\theta_i \geq \sum_{\substack{k=1 \\ k \neq i}}^n \omega_{ki} \theta_k. \quad \triangle$$

Because  $\theta_i = \frac{\nu_i}{\alpha_i}$ , we can regard  $\theta_i$  as measure of how social agent  $i$  is. Assumption 3 then requires that agent  $i$ 's outgoing influence, measured upon agent  $k$  through  $\omega_{ki}$ , be limited based on its own sociability. To influence other highly social agents, agent  $i$  must itself be more social. Conversely, if agent  $i$  is less social, then it will have a smaller value of  $\theta_i$ , and the weights of its outgoing edges must be smaller.

Computing the equilibrium of these dynamics, we set  $\dot{\gamma} = 0$ , which gives  $\gamma_0 = \ln(1 - \mathbb{1}^T y_0)$ . To find  $y_0$ , we have  $\dot{y} = 0 = B(A\mathbb{1}e^{\gamma_0} - A\rho - VTy_0)$  where solving for  $y_0$  gives  $y_0 = (A\mathbb{1}\mathbb{1}^T + VT)^{-1} A(\mathbb{1} - \rho)$ , which is well-defined under Assumption 1.

To shift this system's equilibrium to the origin, we then define the states  $v = \gamma - \gamma_0$  and  $w = y - y_0$ . Computing  $\dot{v}$  and  $\dot{w}$ , we find

$$\begin{aligned} \dot{v} &= -e^{\gamma_0} (e^v - 1) - \mathbb{1}^T w \\ \dot{w} &= e^{\gamma_0} BA\mathbb{1}(e^v - 1) - BVTw. \end{aligned} \quad (2a)$$

### 3.2 Global Asymptotic Stability

The following theorem shows that the system model in Equation (2) is globally asymptotically stable. Due to its technical nature, we relegate the proof to the appendix.

**Theorem 1.** The system in Equation (2) is globally asymptotically stable to the origin.

*Proof:* See the Appendix. ■

While stability is of course critical to insuring acceptable asymptotic network behaviors, stability as a purely dynamical systems notion does not account for the fact that this is a resource consumption network. The value of the resource itself must be sustained over time, and thus its dynamics and those of the agents' consumptions must be accounted for not only in the limit, but also at each point in time as the system evolves. Accordingly, we next assess the role of sustainability in this system.

## 4 FORMALIZING SUSTAINABILITY

In the quest for attaining sustainable societies, one of the biggest challenges is formalizing a notion of sustainability. The Brundtland Commission of the WCED [2] defines sustainable development as "development that meets the needs of the present without compromising the ability of future generations to meet their own needs". Clearly, this definition leaves much room for interpretation, and a vast amount of literature is devoted to developing a definitive measure of sustainability. As a result, there exist varying notions of sustainability originating from different disciplines ranging from the ecological sciences [3] to economics [25]. These discrepancies create significant difficulty in characterizing sustainable behavior in the rigorous framework of dynamical systems.

It is quite natural to think of sustainable consumption in social-ecological systems as a state in which the level of consumption is allowed to be non-decreasing [26]. On the other hand, it is broadly understood that a continuously increasing consumption is not realizable in a finite world with limited resources [27]. Sustainability would then imply an indefinitely maintainable level of consumption, which corresponds to attaining a positive equilibrium. It is not surprising therefore, that stability has been closely linked with sustainability by researchers working in the area (see for instance [6], [7], [8]).

However, while stability is roughly understood to be a necessary property for sustainability, it is not sufficient [7]. In particular, from an economical viewpoint, Chichilnisky [4] argues that sustainability should not only rely on the behavior of the consumption path in the limit (as time approaches infinity), but also on the accumulated utility for all time. From an

ecological perspective, an ecosystem can remain functional only if its critical parameters remain within a certain region over time [28], which likewise suggests the use of some non-asymptotic analysis.

Martinet formalizes this phenomenon in his definition for sustainability as bounds on certain sustainability indicators [5]. He then defines a sustainability criterion as a generalized maximin problem in search of the bounds that are optimal with respect to certain preference functions. The boundedness of system response is captured by the concept of Lagrange stability in dynamical systems theory [29]. Lagrange stability has also been related in the ecological literature with the ability of an ecosystem to resist disturbances [7]. However Lagrange stability (along with other classical notions of stability [30]) is an absolute property of the system rather than its trajectories. This poses a challenge in prioritizing different trajectories based on their level of sustainability. In other words, stability alone does not capture the nuances of sustainability.

Beyond stability, another property of the system that is closely linked with sustainability is resilience. Holling [31] defines resilience as the ability of a system to maintain its integrity when subjected to disturbances. From a dynamical systems viewpoint, this property can be related to how rapidly or gradually a system moves towards an equilibrium. For social-ecological systems, a sudden change in consumption patterns may trigger a major regime change in the underlying habitat accommodating the natural resource, which in turn may cause unpredictable and undesirable changes in the resource stock itself. Thus, a consumption path must not vary at more than a certain rate for it to be sustainable. The notion of sustainability as permitting a non-decreasing consumption path already implies a lower bound on the derivative of the consumption trajectory [32]. However the above argument also implies an upper bound on the derivative. It is important to note that while a limit on the rate of change of the trajectories protects the system from loss of resilience, it also further reflects the physical limits associated with the rate of resource growth and harvesting.

Even across the different notions of sustainability present in the literature, e.g., [26], [32], [33], [34], the underlying philosophy remains the same: sustainability entails preserving essential system characteristics over a sufficiently long period of time. What distinguishes these notions then, is the different perspectives on what characteristics of the system must persist and for how long. Costanza and Patten [7] assert that sustainability is highly dependent on the scale at which the system is perceived. In particular, the sustainability of a larger system does not imply the sustainability of all of its subsystems. For example, cells and bacteria die periodically in order for a larger

organism to sustain its life, or a small city may be consumed by construction of a dam in order to promote the sustainability of the region as a whole. The losses of these smaller subsystems prolong the sustainability of the larger systems, though it is broadly understood that this does not provide “maintenance forever” of the larger system [35]. Sustainability thus cannot imply an infinite lifetime. Rather, it implies that the system achieves its expected lifespan, and this lifespan depends both on the temporal and spatial scales of the system of interest [35].

In light of the above discussion, it is possible to identify the salient features that we can use to mathematically encode sustainability. These are listed in sequence below:

- First, sustainability should be evaluated over finite intervals. This reflects that sustainability of a system does not necessarily mean indefinite survival, but rather the attainment of its expected lifespan [7]. As discussed above, this lifespan depends on the natural temporal and spatial scales of a system.
- Second, sustainability should provide criteria pointwise in time as opposed to only at the steady-state behavior of a system. Sustainability concerns itself not only with the generation in the time limit, but also with the current generation and all others in between [2]. Thus, rather than being asymptotic, a definition of sustainability should account for all states of a system over time [4], [26].
- Third, sustainability should be a structural property of a system [28]. From a systems perspective, the structure of the system determines the characteristics that are essential to attaining a particular state of being. In this context, sustainability may be thought of as the invariance of a particular quantity (mediated by the internal structure) that captures the required behavior. The model in Section 2 is a networked system, and the “structure” of this system consists of both its topology and associated edge weights, which are encapsulated by the matrix  $T$ .

Following the guidelines listed above, we develop a mathematical formulation of sustainability as follows. First, we consider sustainability for systems across the time horizon  $[0, t_{max}]$  for some  $t_{max} > 0$ . Second, we define the following four constants:

- 1)  $v_{max}$ , the maximum allowable value of  $v$
- 2)  $v_{min}$ , the minimum allowable value of  $v$
- 3)  $d_{max}$ , the maximum allowable value of  $\dot{v}$
- 4)  $d_{min}$ , the minimum allowable value of  $\dot{v}$ ,

where  $d$  is chosen to indicate “derivative”.

Together, the four constants listed above define a box in the  $(v, \dot{v})$ -plane, and sustainability of a system is equivalent to invariance of this box. We emphasize that the equilibrium of  $v$  need not be in this box, which further highlights the distinction between stability and sustainability. The above criteria are motivated by purely ecological concerns, though they naturally lead us to consider forward invariance of a compact region of the system’s state space. In particular, we define the subset  $S$  of the  $(v, \dot{v})$ -plane as

$$S := \{ (v, \dot{v}) \mid v \geq v_{min}, v \leq v_{max}, \dot{v} \geq d_{min}, \dot{v} \leq d_{max} \},$$

and sustainability for a social-ecological system is then defined as follows.

**Definition 1.** A quantity of resource  $v(t)$  is sustainable over  $[0, t_{max}]$  if  $(v(t), \dot{v}(t)) \in S$  for all  $t \in [0, t_{max}]$ .  $\triangle$

Note that many of the above sustainability notions from the existing literature pertain directly to consumption, while our mathematical sustainability criterion is stated in terms of the resource. Despite this apparent difference, we show in Section 5 below that Definition 1 must account for agents’ consumptions to be satisfied. In particular, we show how Definition 1 is linked with both the social structure and individual parameters of the consuming network by deriving conditions on  $T$ ,  $A$  and  $B$  that are sufficient to satisfy Definition 1.

## 5 INSURING SUSTAINABILITY OF NETWORKED RESOURCE CONSUMPTION

The social structure of a resource consumption network is specified both by the graph of agents’ interactions and the weights of edges in this graph. These values are captured in the matrix  $T$  in Equation (2), and this section will derive conditions on  $T$  that insure that  $v$  remains at sustainable levels over some pre-specified time horizon  $[0, t_{max}]$ . These results require not only accounting for how  $v$  evolves, but also how agents’ consumption levels affect  $v$ .

### 5.1 Preliminaries

Towards deriving the desired conditions on  $T$ , we first give some preliminary results related to  $v$  and  $w$  that we will use below. We begin by stating the Bellman-Grönwall inequality in the required form.

**Lemma 1. Bellman-Grönwall Inequality; [36])** Let  $K_1 \in \mathbb{R}$  and  $K_2 > 0$ . If, for all  $t \in [a, b]$ ,  $\phi : [a, b] \rightarrow \mathbb{R}$ , the bound  $\phi(t) \leq K_1 + K_2 \int_a^t \phi(s) ds$  is satisfied, then on the same interval  $\phi(t) \leq K_1 e^{K_2(t-a)}$ .  $\blacksquare$

Next, we enforce the following assumption.

**Assumption 4.** We assume that

- $v_{min} < v_{max}$
- $v_{max} > 0$
- $v_{min} < v(0) < v_{max}$
- $d_{min} < 0 < d_{max}$ .  $\diamond$

This assumption is rather mild in what it imposes upon the system. Requiring that  $v_{max} > 0$  and  $v_{max} > v_{min}$  insures that a given set of sustainability bounds is feasible. Assuming that  $v_{min} < v(0) < v_{max}$  merely assumes that a system starts within the bounds it must remain within. And requiring that  $d_{min} < 0 < d_{max}$  means that the resource levels in a system should be allowed to both increase and decrease, which is natural.

Next, we define the constants  $\beta = \max_{i \in \{1, \dots, n\}} b_i \nu_i$ ,  $C_1 = \|w(0)\|_1 + t_{max} e^{\gamma_0} (e^{v_{max}} - 1) \sum_{i=1}^n b_i \alpha_i$ , and  $C_2 = \beta \|T\|_1 t_{max}$ , and we will repeatedly encounter them below. Using Lemma 1 and Assumption 4, we have the following characterization of  $w$ .

**Lemma 2.** Suppose that  $v(s) \leq v_{max}$  for all  $s \in [0, t]$ . Then  $\|w(t)\|_1 \leq C_1 \exp(C_2)$ .

*Proof:* The fundamental theorem of calculus gives

$$w(t) = w(0) + \int_0^t e^{\gamma_0} B A \mathbb{1} (e^v - 1) - B V T w(\tau) d\tau.$$

Taking the 1-norm of both sides and applying the triangle inequality then gives

$$\begin{aligned} \|w(t)\|_1 &\leq \|w(0)\|_1 + \int_0^t \|e^{\gamma_0} B A \mathbb{1} (e^{v(\tau)} - 1)\|_1 d\tau \\ &\quad + \int_0^t \|B V\|_1 \|T\|_1 \|w(\tau)\|_1 d\tau \\ &\leq \|w(0)\|_1 + t_{max} e^{\gamma_0} \|B A \mathbb{1} (e^{v_{max}} - 1)\|_1 \\ &\quad + \|B V\|_1 \|T\|_1 \int_0^t \|w(\tau)\|_1 d\tau \\ &= \|w(0)\|_1 + t_{max} e^{\gamma_0} (e^{v_{max}} - 1) \sum_{i=1}^n b_i \alpha_i \\ &\quad + \|B V\|_1 \|T\|_1 \int_0^t \|w(\tau)\|_1 d\tau, \end{aligned}$$

where we have used that  $v(\tau) \leq v_{max}$  and expanded the 1-norm of  $B A \mathbb{1}$ .

Using that the 1-norm of a matrix is equal to its largest column sum, the value of  $\|B V\|_1$  is then equal to  $\beta$  because  $B V$  is diagonal. The lemma follows by applying Lemma 1 with  $K_1 = C_1$  and  $K_2 = C_2$ .  $\blacksquare$

We next derive a similar bound for  $v$ .

**Lemma 3.** Suppose that  $v(s) \leq v_{max}$  for all  $s \in [0, t]$ .

Under the dynamics in Equation (2), we find that

$$v(t) \geq v(0) - t_{max} [e^{\gamma_0} (e^{v_{max}} - 1) - C_1 \exp(C_2)].$$

*Proof:* From the fundamental theorem of calculus,

$$v(t) = v(0) - \int_0^t e^{\gamma_0} (e^{v(s)} - 1) ds - \int_0^t \mathbb{1}^T w(s) ds.$$

Noting that  $e^{v(\tau)} - 1 \leq e^{v_{max}} - 1$ , we find

$$v(t) \geq v(0) - t e^{\gamma_0} (e^{v_{max}} - 1) - \int_0^t \mathbb{1}^T w(\tau) d\tau.$$

For the final term above we find that

$$\int_0^t \mathbb{1}^T w(\tau) d\tau \leq \int_0^t \|w(\tau)\|_1 d\tau.$$

Multiplying by  $-1$  to reverse the inequality, we find

$$v(t) \geq v(0) - t e^{\gamma_0} (e^{v_{max}} - 1) - \int_0^t \|w(\tau)\|_1 d\tau.$$

We complete the proof by applying Lemma 2.  $\blacksquare$

Having established these basic preliminary lemmas, we next derive bounds on  $T$  to enforce the sustainability bounds on  $v$  and  $\dot{v}$ .

## 5.2 Enforcing $v(t) \leq v_{max}$

To enforce  $v(t) \leq v_{max}$ , we enforce  $\dot{v}|_{v=v_{max}} \leq 0$ , and the following lemma gives a sufficient condition for doing so.

**Lemma 4.** Under Assumption 4, if  $T$  satisfies

$$\|T\|_1 \leq \frac{1}{\beta t_{max}} \ln \left( \frac{e^{\gamma_0} (e^{v_{max}} - 1)}{C_1} \right)$$

then  $v(t) \leq v_{max}$  for all  $t \in [0, t_{max}]$ .

*Proof:* Using Equation (2a), enforcing  $\dot{v}|_{v=v_{max}} \leq 0$  is equivalent to requiring  $-e^{\gamma_0} (e^{v_{max}} - 1) - \mathbb{1}^T w \leq 0$ , which we rearrange to find  $-\mathbb{1}^T w \leq e^{\gamma_0} (e^{v_{max}} - 1)$ .

We note that  $-\mathbb{1}^T w \leq \|w\|_1$ , and we will thus enforce the sufficient condition

$$\|w(t)\|_1 \leq e^{\gamma_0} (e^{v_{max}} - 1) \quad (3)$$

for all  $t$ . Using Assumption 4,  $v(0) < v_{max}$ . Suppose that  $v$  approaches  $v_{max}$  at time  $t$ , i.e.,  $v(t) = v_{max}$ . Then the condition  $v(s) \leq v_{max}$  for all  $s \in [0, t]$  is satisfied and Lemma 2 can be used at time  $t$ . Then a sufficient condition for enforcing Equation (3) is given by

$$C_1 \exp(\beta \|T\|_1 t_{max}) \leq e^{\gamma_0} (e^{v_{max}} - 1).$$

Solving for  $\|T\|_1$  then gives a bound that insures  $\dot{v}(t)|_{v=v_{max}} \leq 0$ . If  $v$  later approaches  $v_{max}$  at some time  $\tau$ , then repeating the above argument at  $\tau$  insures that  $\dot{v}(\tau)|_{v=v_{max}} \leq 0$  as well. Of course, if  $v$  never approaches  $v_{max}$  then  $v(t) \leq v_{max}$  trivially holds, and thus it is true for all  $t \in [0, t_{max}]$  in all cases.  $\blacksquare$

### 5.3 Enforcing $v(t) \geq v_{min}$

We next derive a sufficient condition on  $T$  that implies that the bound  $v(t) \geq v_{min}$  is satisfied for all  $t \in [0, t_{max}]$ .

**Lemma 5.** If  $\|T\|_1$  satisfies the bound in Lemma 4 and

$$\|T\|_1 \leq \frac{1}{\beta t_{max}} \ln \left( \frac{v(0) - t_{max} e^{\gamma_0} (e^{v_{max}} - 1) - v_{min}}{t_{max} C_1} \right), \quad \|T\|_1 \leq \frac{1}{\beta t_{max}} \ln \left( \frac{|d_{min}| - e^{\gamma_0} (e^{v_{max}} - 1)}{C_1} \right),$$

then  $v(t) \geq v_{min}$  for all  $t \in [0, t_{max}]$ .

*Proof:* The bound from Lemma 4 insures that  $v(t) \leq v_{max}$  for all  $t \in [0, t_{max}]$  and thus Lemma 3 can be applied. Using Lemma 3, a sufficient condition for this lemma is to enforce

$$v(0) - t_{max} e^{\gamma_0} (e^{v_{max}} - 1) - t_{max} C_1 \exp(C_2) \geq v_{min}.$$

Rearranging terms and expanding  $C_2$  we find

$$\begin{aligned} v(0) - t_{max} e^{\gamma_0} (e^{v_{max}} - 1) - v_{min} \\ \geq t_{max} C_1 \exp(\beta \|T\|_1 t_{max}), \end{aligned}$$

where solving for  $\|T\|_1$  completes the proof.  $\blacksquare$

With Assumption 4 in place, Lemmas 4 and 5 jointly provide that  $v(t) \in [v_{min}, v_{max}]$  for all  $t \in [0, t_{max}]$ , and the next two lemmas rely on this point.

### 5.4 Enforcing $\dot{v} \leq d_{max}$

In this section we derive a condition on  $\|T\|_1$  that implies  $\dot{v}(t) \leq d_{max}$  for all  $t \in [0, t_{max}]$ .

**Lemma 6.** Suppose that  $\|T\|_1$  satisfies the bound in Lemma 4. If  $\|T\|_1$  also satisfies

$$\|T\|_1 \leq \frac{1}{\beta t_{max}} \ln \left( \frac{d_{max} + e^{\gamma_0} (e^{v_{min}} - 1)}{C_1} \right)$$

then  $\dot{v}(t) \leq d_{max}$  for all  $t \in [0, t_{max}]$ .

*Proof:* Similar to above, we use that

$$\dot{v} = -e^{\gamma_0} (e^v - 1) - \mathbb{1}^T w \leq -e^{\gamma_0} (e^{v_{min}} - 1) + \|w(t)\|_1$$

to derive a sufficient condition. In particular, because  $T$  satisfies the bound in Lemma 4 we may apply Lemma 2. Doing so, we wish to enforce

$$-e^{\gamma_0} (e^{v_{min}} - 1) + C_1 \exp(C_2) \leq d_{max}.$$

Rearranging we find

$$C_1 \exp(\beta \|T\|_1 t_{max}) \leq d_{max} + e^{\gamma_0} (e^{v_{min}} - 1),$$

where isolating  $\|T\|_1$  completes the proof.  $\blacksquare$

### 5.5 Enforcing $\dot{v} \geq d_{min}$

This section derives a sufficient condition on  $\|T\|_1$  that implies  $\dot{v} \geq d_{min}$  for all  $t \in [0, t_{max}]$ .

**Lemma 7.** Suppose that  $T$  obeys the bound in Lemma 4. If  $\|T\|_1$  also obeys the bound

$$\|T\|_1 \leq \frac{1}{\beta t_{max}} \ln \left( \frac{|d_{min}| - e^{\gamma_0} (e^{v_{max}} - 1)}{C_1} \right),$$

then  $\dot{v}(t) \geq d_{min}$  for all  $t \in [0, t_{max}]$ .

*Proof:* Using Equation (2a), a sufficient condition for  $\dot{v}(t) \geq d_{min}$  is  $d_{min} \leq -e^{\gamma_0} (e^v - 1) - \mathbb{1}^T w$ . Rearranging, we wish to enforce  $e^{\gamma_0} (e^v - 1) + \mathbb{1}^T w \leq |d_{min}|$ , where the absolute value comes from the fact that  $d_{min} < 0$  in Assumption 4. A sufficient condition for doing so is  $e^{\gamma_0} (e^v - 1) + \|w(t)\|_1 \leq |d_{min}|$ . By hypothesis,  $T$  satisfies the bound in Lemma 4 and, as a result,  $v(t) \leq v_{max}$  for all  $t \in [0, t_{max}]$ . Then we apply Lemma 2 to find the bound

$$e^{\gamma_0} (e^{v_{max}} - 1) + C_1 \exp(\beta \|T\|_1 t_{max}) \leq |d_{min}|.$$

Solving for  $\|T\|_1$  then gives the result.  $\blacksquare$

### 5.6 Overall Sustainability Bound

Enforcing sustainability requires that  $v(t) \in [v_{min}, v_{max}]$  and that  $\dot{v}(t) \in [d_{min}, d_{max}]$  for all  $t \in [0, t_{max}]$ . Insuring satisfaction of these conditions can be done by satisfying all four of the preceding bounds on  $\|T\|_1$ . Combining the four preceding lemmas, we have the following theorem that states a single unified sustainability criterion.

**Theorem 2.** Suppose that Assumption 4 holds and define the constants

$$\begin{aligned} \xi_1 &= e^{\gamma_0} (e^{v_{max}} - 1) \\ \xi_2 &= \frac{v(0) - t_{max} e^{\gamma_0} (e^{v_{max}} - 1) - v_{min}}{t_{max}} \\ \xi_3 &= d_{max} + e^{\gamma_0} (e^{v_{min}} - 1) \\ \xi_4 &= -d_{min} - e^{\gamma_0} (e^{v_{max}} - 1). \end{aligned}$$

If, for all  $i \in \{1, 2, 3, 4\}$ ,  $\xi_i > C_1$ , then the system in Equation (2) is sustainable for any  $T$  satisfying

$$\|T\|_1 \leq \frac{1}{\beta t_{max}} \ln \left( \frac{\min_{i \in \{1, 2, 3, 4\}} \xi_i}{C_1} \right). \quad (4)$$

The condition that  $\xi_i > C_1$  is simply a feasibility condition; if it is violated, then these bounds cannot be used to insure sustainability because the logarithm will output a negative value. This feasibility condition can be used to generate conditions that must be satisfied by the parameters  $v_{min}, v_{max}, d_{min}, d_{max}$ , and  $t_{max}$ ; we avoid a lengthy exposition on this subject because the many parameters in this model generate a

vast number of possible relationships that one could explore. Instead, we comment on one relationship in particular. Elementary operations show that the condition  $\xi_1 > C_1$  implies that  $\frac{1}{\sum_{i=1}^n b_i \alpha_i} > t_{max}$ . This relationship points to the importance of agent sensitivity, captured by  $b_i$ , when ensuring sustainability. We expand on the relationship between sustainability and sensitivity in the next section, and we further discuss the implications of sustainability in that context.

## 6 SIMULATION AND DISCUSSION

In this section we explore the implications of the above results for the behavior of social-ecological systems. First, the behavior of the model is examined under different  $\theta$  parameters in order to demonstrate the differences between societies that place more weight on social information and societies that place more weight on ecological information; these cases will be referred to as *pro-social* and *pro-ecological* societies, respectively. We will benchmark these cases against societies which weight social and ecological information comparably, which we refer to as the *equal* case. Second, the sustainability bounds are discussed with respect to the parameter  $b_i$ , demonstrating how the sensitivity of populations relates to the sustainability of their consumptions.

### 6.1 Pro-Social and Pro-Ecological Communities

The stability proof presented in Theorem 1 depends on Assumption 3, i.e., that  $\theta_i \geq \sum_{k=1, k \neq i}^n \omega_{ki} \theta_k$  for all  $i$ .

If the collection  $\{\theta_i\}_{i \in \{1, \dots, n\}}$  satisfies Assumption 3, then  $\{\delta \theta_i\}_{i \in \{1, \dots, n\}}$  for  $\delta \geq 0$  also does. While the relationship between the individual  $\theta_i$ 's is fixed by Assumption 3, the scaling factor  $\delta$  allows a variety of systems to be described based on a single collection  $\{\theta_i\}_{i \in \{1, \dots, n\}}$  that satisfies Assumption 3. With such a collection, and using  $\theta_i = \frac{\nu_i}{\alpha_i}$  and  $\nu_i + \alpha_i = 1$ , we set

$$\alpha_i = \frac{1}{1 + \delta \theta_i} \quad \text{and} \quad \nu_i = \frac{\delta \theta_i}{1 + \delta \theta_i}$$

to find new sets of admissible parameters. If  $\delta$  is chosen so that  $\delta \theta_i \gg 1$ , then  $\alpha_i \approx 0$  and  $\nu_i \approx 1$ , which we refer to as the *pro-social* condition. If  $\delta \theta_i \ll 1$ , then  $\nu_i \approx 0$  and  $\alpha_i \approx 1$ , which we refer to as the *pro-ecological* condition. We refer to the case of  $\alpha_i \approx \nu_i$  as the *equal* condition.

To understand the impact of the relative sizes of  $\alpha_i$  and  $\nu_i$  on system behavior, the system in Equation (2) was simulated on the 25-node random graph shown in Figure 1. Each edge was given uniform weight. The choice of  $\theta = [0.1826 \ 0.3296 \ 0.2313 \ 0.3454 \ 0.1987 \ 0.1923 \ 0.1642 \ 0.1989 \ 0.1182 \ 0.2198 \ 0.1124 \ 0.0734 \ 0.1592 \ 0.3608 \ 0.1913 \ 0.1810 \ 0.2098 \ 0.1206 \ 0.3210 \ 0.0606 \ 0.0597$

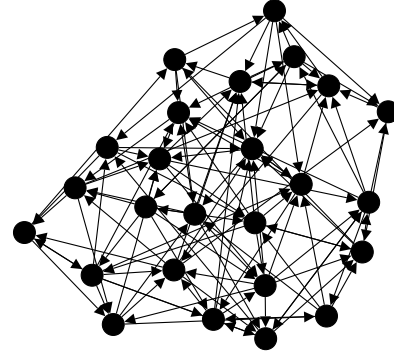


Fig. 1. The 25-node, 114-edge random graph used for simulation visualized with Gephi [37]. Each node in this graph represents an agent and each edge represents that two agents interact.

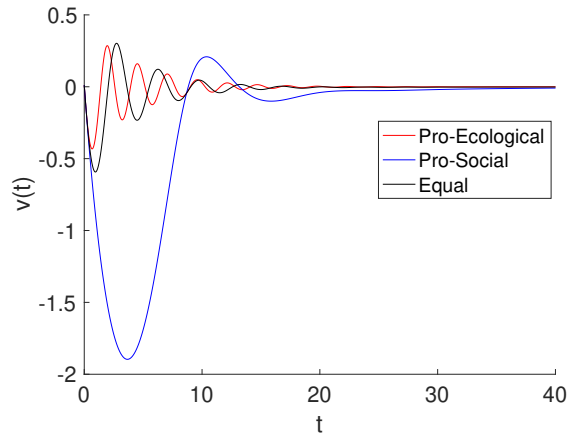


Fig. 2. The behavior of the resource stock  $v(t)$  for the pro-ecological, pro-social, and equal cases when simulated on the network in Figure 1. The pro-ecological and equal cases show similar oscillatory behaviors while the pro-social cases shows lower-frequency oscillations of substantially larger magnitude.

$[0.1302 \ 0.0808 \ 0.1336 \ 0.1638]^T$  insures satisfaction of Assumption 3. Three cases were run to elucidate the impacts of varying  $\alpha_i$  and  $\nu_i$ . In terms of their respective averages,  $\bar{\alpha}$  and  $\bar{\nu}$ , these cases are:

- i. The pro-ecological case with  $\delta = 0.1$ , which gives  $\bar{\alpha} = 0.9822$  and  $\bar{\nu} = 0.0196$
- ii. The pro-social case with  $\delta = 10$ , which gives  $\bar{\alpha} = 0.0644$  and  $\bar{\nu} = 0.9356$
- iii. The *equal* case with  $\delta = \theta = 0.1816$ , which gives  $\bar{\alpha} = 0.5261$  and  $\bar{\nu} = 0.4739$ .

The parameters  $b_i$  and  $\rho_i$  were chosen uniformly at random from  $[0, 1]$ .

The resource behavior for all three runs is shown in Figure 2 and the individual consumption behavior for all three runs is shown in Figure 3; each plot is representative of the behavior observed on a variety of graph topologies over the course of many simulations. We see that both the resource stock and consumption



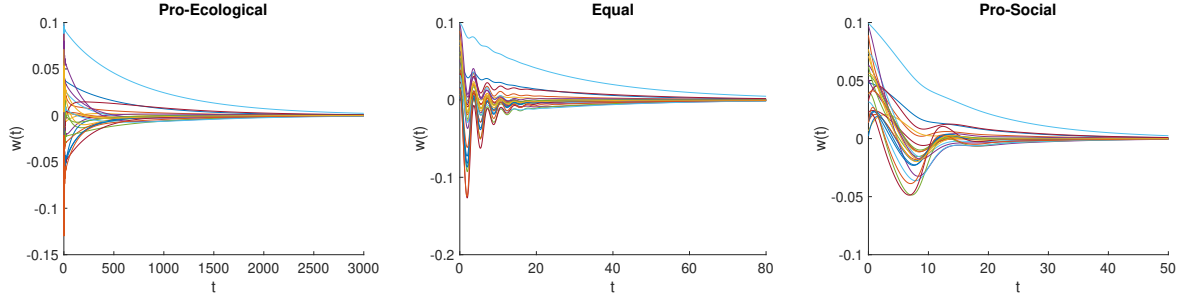


Fig. 3. The individual consumptions,  $w_i(t)$ ,  $i \in \{1, \dots, 25\}$ , for the aforementioned 25-node graph under the pro-social, pro-ecological, and equal conditions. It takes approximately 2000 time steps for the pro-ecological network to converge, compared with 100 and 50 time steps for the equal and pro-social networks, respectively.

values tend to fluctuate to varying degrees, depicted by the oscillations in these trajectories. Such oscillations may be undesirable in social-ecological systems for a variety of reasons. For instance, it has been observed that grasslands and forests exhibit an increased tendency to be eradicated by invading plant species in the presence of frequent disturbances and repeated exposure to extreme environmental conditions [38], [39]. In addition, boom-bust cycles in human-natural systems may not only lead to undesirable development patterns in the linked communities [40], but also unwanted price fluctuations by means of substantial supply shocks [41].

Among the three shown simulation runs, Figure 2 shows that the pro-social system exhibits relatively fewer total oscillations than the other systems, though the magnitude of oscillations is larger. Note that in Figure 2, while the resource dynamics for the pro-ecological and equal societies display significantly more fluctuations than the pro-social society, they converge to the equilibrium faster. The notion of speed of convergence, and of return to the equilibrium, have been previously associated with the concept of resilience, a fundamental attribute of sustainable systems [7], [42]. We then see from Figure 2 that while the pro-social resource dynamics have desirably fewer fluctuations, the pro-ecological dynamics have a desirably higher rate of convergence with smaller magnitudes of oscillations. The situation is reversed if we observe the trajectories of the individual consumptions in Figure 3. Here, the pro-ecological society exhibits fewer oscillations but with slower convergence, while the pro-social society exhibits faster convergence but with more oscillations.

There thus exist two trade-offs. The first is between fluctuations and rate of convergence, with faster convergence coming at the cost of more fluctuations in the resource, and reduced fluctuations coming at the cost of slower convergence to an equilibrium value. The second tradeoff is between desirable behaviors in the resource and desirable behaviors in agents' consump-

tion levels, with reduced fluctuations in each coming at the cost of increased fluctuations in the other. These tradeoffs suggest that the equal society, which gives a balanced preference to both ecological and social information, is more favorable than extreme societal configurations that weigh one source of information much more heavily than the other. From a policy perspective, promoting the equal society would entail measures that encourage society to give equal consideration to the state of the resource and the behavior of neighboring agents while determining individual consumptions. Examples of such measures include strategic information dispersion, targeted advertising, awareness campaigns, and manipulating visibility of key variables [43].

## 6.2 System Sensitivity

The conditions required by Theorem 1 did not restrict the parameters  $\{b_i\}_{i \in \{1, \dots, n\}}$ , which factors out of the consumption dynamics for  $\dot{w}$  and thus essentially acts as a gain on agents' responsiveness to network stimuli. That is, the parameter  $b_i$  captures agent  $i$ 's sensitivity to the social and ecological information they receive. However, the sustainability bounds derived in Section 4 depend upon  $\{b_i\}_{i \in \{1, \dots, n\}}$  through the parameter  $\beta$  and the term  $\sum_{i=1}^n b_i \alpha_i$  in the definition of  $C_1$ . Furthermore, as discussed previously, for  $\xi_1 > C_1$  it must hold that  $\frac{1}{\sum_{i=1}^n b_i \alpha_i} > t_{max}$ , implying that agents' sensitivities affect bounds on the time horizon over which sustainability can be guaranteed.

We consider the implications of this bound in the same 25-node network shown in Figure 1 under the equal parameter regime. We consider three uniform cases:  $b_i = 0.025$ ,  $b_i = 0.005$ , or  $b_i = 0.001$  for all  $i$ , and changing  $b_i$  changes the time horizon over which a system can be shown to be sustainable. We consider a time horizon of  $t_{max} = 1$  when  $b_i = 0.025$ , a time horizon of  $t_{max} = 2$  when  $b_i = 0.005$ , and a time horizon of  $t_{max} = 3$  when  $b_i = 0.001$ . Note that these times are normalized with respect to the intrinsic

growth rate of the resource, and thus sustainability for  $t_{max} = 3$  pertains to the system behavior across an interval that is 3 times the intrinsic time constant of the resource [22].

The bounds in Equation (4) determine sustainability with respect to choices of  $v_{min}, v_{max}, d_{min},$  and  $d_{max}$ , which would be determined by the ecological context that is being modeled. For this simulation, we consider the minimal  $v_{max}$  and  $d_{max}$  as well as the maximal  $v_{min}$  and  $d_{min}$  for which the system is sustainable, which we refer to as the *minimal sustainability window*. These values can be derived from elementary operations on the condition that  $\xi_i = e^{\|T\|_1 \beta t_{max}} C_1$  for  $i \in \{1, 2, 3, 4\}$ , which follows from attaining equality in the bound in Theorem 2. We plot the evolution of the resource from  $t = 0$  to  $t = t_{max}$  in Figure 4, where we see that it indeed remains within the minimal sustainability window. In Figure 4, as  $b_i$  decreases the minimal sustainability window more tightly bounds the resource trajectory, suggesting that the resulting system is sustainable with respect to a larger set of choices of  $v_{min}, v_{max}, d_{min},$  and  $d_{max}$ . This also suggests that a decrease in agent sensitivity results in longer time horizons for which the system can be made sustainable.

However, sensitive systems have been observed to be more beneficial in certain settings and with respect to a particular set of indicators. For instance, Consumer Affect (intensity of reaction to stimuli [44]) has been found to have a profound effect on achieving sustainable consumer behavior [43]. The ability of agents to adapt quickly to resource fluctuations has also been found to be an important aspect of sustainable fishing communities [45]. We find in our model that low sensitivity can indeed be associated with higher fluctuations in the resource.

However, as highlighted in the preceding sections, social-ecological systems are complex in nature, where a single factor alone cannot be associated with a particular outcome. Instead, system parameters and variables often act in combination to produce a certain outcome. To further illustrate this point, Figure 5 shows the resource trajectory for a relatively less-sensitive society. The simulation was run for the same 25-node graph as above, with  $b_i = 0.005$  and for all configurations of information preference used above. The pro-social society has a very large fluctuation in the quantity of the resource, reaching<sup>1</sup>  $v \approx -30$ , which is roughly 6 times the size of fluctuations in the equal society and nearly 10 times that of the pro-ecological one. This suggests that insufficient weighting of ecological information and low sensitivity, together, lead to large, potentially harmful fluctuations

1. The negative value here is due to the logarithmic coordinate transformation to arrive at the system in  $(v, w)$ -coordinates.

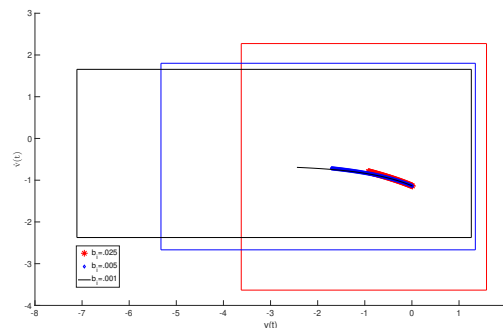


Fig. 4. The resource,  $v(t)$ , versus  $\dot{v}(t)$  from  $t = 0$  to  $t = t_{max}$  with  $b_i = 0.025$  for all  $i$ ,  $b_i = 0.005$  for all  $i$ , and  $b_i = 0.001$  for all  $i$  on the network in Figure 1 with  $\alpha_i = \nu_i$  for all  $i$ . The red, blue, and black boxes are the minimal sustainability window for  $b_i = 0.025$ ,  $b_i = 0.005$ , and  $b_i = 0.001$ , respectively. As  $b_i$  decreases, the trajectory elongates, reflecting both a larger  $t_{max}$  and the fact that the minimal sustainability window more tightly bounds the trajectories.

in the resource. This point has been observed previously in [46], where more sensitivity to ecological information has been observed to contribute to adopting more sustainable behavior patterns.

The above suggests another trade-off enacted by varying the sensitivity of the agents. On one hand, higher sensitivity leads to loose bounds and small time horizons for the sustainability criterion. On the other hand, low sensitivity leads to larger oscillations in the resource stock (especially in combination with low preference for ecological information). The model hence suggests, from a policy perspective, controlled measures to vary the responsiveness of the consuming population. These measures may either be of an informational or structural nature (see [43] and included references for possible examples of such measures), with the end goal always being improving sustainability outcomes.

## 7 CONCLUSION

In this paper we have presented a mathematical criterion for sustainability grounded in the literature on sustainable development. The presented definition of sustainability extends the notion of stability which is often applied to ecological systems. While stability pertains to the asymptotic behavior of the system, sustainability concerns itself with behavior over a finite time horizon where transient behavior often dominates. We find that, in the chosen model of natural resource consumption, the requirements for sustainability are not captured by stability alone. In particular, we observe that agent sensitivity, while having no contribution to determining stability, has a major role in sustainability of the same system. Our

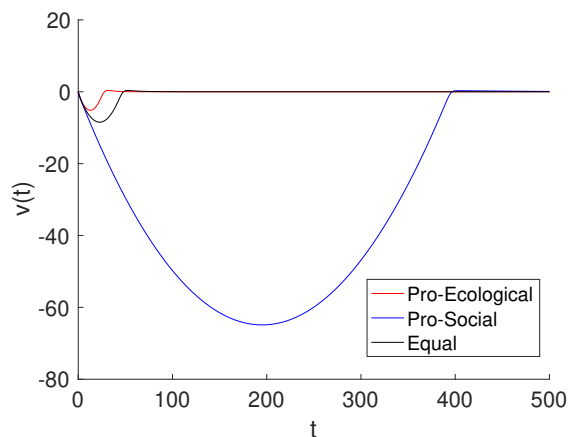


Fig. 5. The behavior of the resource stock,  $v(t)$ , for the 25-node graph in Figure 1 under three parameter conditions on  $\alpha$  and  $\nu$  when  $b_i = 0.005$  for all  $i$ . Here the pro-social condition has oscillations that are much larger than the other two cases.

simulations of sustainable networks for the model uncover tradeoffs in system behavior that do not appear when investigating system stability alone. These findings may be translated to guidelines for effective policy-making in natural resource systems. This study thus serves as one step towards a dynamical systems theory for sustainability of social-ecological systems.



**Matthew Hale** is an Assistant Professor of Mechanical and Aerospace Engineering at the University of Florida. He received his BSE in Electrical Engineering summa cum laude from the University of Pennsylvania in 2012, and his MS and PhD in Electrical and Computer Engineering from the Georgia Institute of Technology in 2015 and 2017, respectively. He was awarded an institute-wide Sigma Xi Best Dissertation award for his doctoral work. He directs the Control, Optimization, and Robotics Engineering (CORE) Lab at the University of Florida, and his research interests include multi-agent systems, mobile robotics, privacy in control, distributed optimization, and graph theory.



**Sebastian Ruf** received his B.S. and M.S. in Mechanical Engineering from the University of California, Berkeley in 2011 and 2012, respectively, and his Ph.D. in Electrical and Computer Engineering from the Georgia Institute of Technology in 2018. He is currently a postdoctoral researcher in the Physics and Psychology Department at Northeastern University on Boston, MA. His research interests include modeling and control of complex networked systems, including socio-ecological and biological systems.



**Talha Manzoor** completed his PhD in Electrical Engineering from LUMS, Lahore in 2017. Prior to this, he received an MS in Computer Engineering from LUMS in 2013 and a BS in Mechatronics Engineering from NUST, Islamabad in 2010. Since January 2018, he is serving as an Assistant Professor in the Electrical Engineering Department at Namal College, Mianwali. Talha is generally interested in the modelling and control of complex Cyber Physical Social Systems. He has also worked on control and estimation problems in robotics which includes both theoretical and developmental work in robotic vision, SLAM and state estimation.



**Abubakr Muhammad** is an associate professor of electrical engineering and the founding director of the Center for Water Informatics & Technology (WIT) at Lahore University of Management Sciences (LUMS) – Pakistans premier private research university. In 2005, he received his Ph.D. in Electrical Engineering Georgia Institute of Technology where his doctoral thesis won a best dissertation award. Before joining LUMS, he was a postdoctoral researcher first at the University of Pennsylvania and then at McGill University. He has won fellowships or held visiting professorial positions at other top institutes including the University of Oxford, ICTP and KAUST. His current interests are at the intersection of environment, technology and society; covering topics related to hydro-informatics, precision agriculture, robotics and impacts of artificial intelligence & automation on society.

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